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SMOOTHNESS OF HOLONOMIES FOR CODIMENSION 1 HYPERBOLIC DYNAMICS

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ABSTRACT

Hyperbolic invariant sets Λ of $C^{1+\gamma}$ diffeomorphisms where either the stable or unstable leaves are 1-dimensional are considered in this paper. Under the assumption that the Λ has local product structure, the authors prove that the holonomies between the 1-dimensional leaves are $C^{1+\alpha}$ for some $0 < \alpha < 1$.

1. Introduction

In this paper we study the laminations by stable and unstable manifolds associated with a hyperbolic invariant set of a $C^{1+\gamma}$ diffeomorphism which is topologically transitive and has local product structure. We consider the codimension 1 case where either the stable or unstable manifolds are 1-dimensional, and prove that the holonomies between the 1-dimensional leaves are $C^{1+\alpha}$ for some $0 < \alpha < 1$. This result (Theorem 2.1) is very useful in a number of contexts. In general terms, it allows one to reduce many questions about 2-dimensional dynamics to simpler questions about 1-dimensional dynamics. In related work, we use it: (i) to construct a Teichmüller space for all C^{1+} conjugacy classes of hyperbolic sets of surface diffeomorphisms with 1-dimensional stable and unstable manifolds [8], (ii) to construct all such systems with an invariant measure with a given geometric measure class (such as all Anosov diffeomorphisms with an invariant measure that is absolutely continuous with respect to 2-dimensional Lebesgue measure) [10], and (iii) to prove that if the basic holonomies of such dynamical systems are sufficiently smooth, then the system is rigid in the sense that it is C^{1+} conjugate to an affine model [9].

Although the results that we prove have been in the folklore for some time, there appears to be no published statement or proof of them. There are a number of results about smoothness of the holonomies of Anosov diffeomorphisms. A crucial component of Hopf's proof of the ergodicity of the geodesic flow of a closed surface of negative curvature was the fact that the holonomies were C^1 . Anosov showed that for general Anosov systems, the stable and unstable foliations are always α -Hölder for some α depending on the rates of expansion and contraction of the system [1]. On the other hand, he knew that the foliations need not be C^2 . Anosov used the fact that the holonomy maps of C^2 Anosov diffeomorphisms have a Hölder continuous Jacobian to show that, when such maps preserve Lebesgue measure, they are ergodic. In the case of codimension 1 Anosov systems, one can use this Jacobian to show that the holonomies are $C^{1+\alpha}$ for some $\alpha > 0$; see [6, Chapter III, Exercise 3.1]. Hirsch and Pugh proved in [3] that if the stable manifolds have codimension one and fill up an open set, then they provide a C^1 foliation of that set.

For more general hyperbolic sets, a number of papers address the question of the regularity of the invariant foliations via the regularity of their tangent distributions. A general result along these lines, showing that the tangent distributions vary in a Hölder continuous way, is given by Hirsh, Pugh and Shub [4, Section (3.8), Remark 1]. As explained in [11], this is not the same as regularity of holonomies. In [12], Schmeling and Siegman-Schulze have proved that the holonomies associated with hyperbolic sets are Hölder.

There are a number of stronger results about restricted classes of Anosov diffeomorphisms. Hirsch and Pugh proved that the foliations are C^1 for area-preserving Anosov diffeomorphisms [3]. Recently, this regularity was improved to $C^{1+\mathcal{O}(t|\log t|)}$ by Hurder and Katok [5] for area-preserving Anosov diffeomorphisms, and by Hasselblatt [2] to C^{2-} for Anosov diffeomorphisms of the torus that satisfy some local spectral conditions.

Our approach is geometric, and applies not only to Anosov systems but also when the hyperbolic set Λ is a proper subset. In this case, the holonomies are defined only on subsets of the local leaves of the laminations. To say that the holonomies are $C^{1+\alpha}$ for some $0 < \alpha < 1$ in this case means that they have a $C^{1+\alpha}$ diffeomorphic extension to the local leaf.

The paper [11] contains a very interesting discussion of different notions of smooth foliation, and gives necessary and sufficient conditions for a $C^{1+\alpha}$ foliation in terms of the smoothness of both the leaves and holonomies plus the variation in the holonomies from leaf to leaf. Unfortunately, these results do not apply directly to our situation, as in our case the invariant leaves may only give a lamination and not a foliation. It would be interesting to extend the discussion in [11] to laminations. However, in the case of hyperbolic sets on surfaces, we are able to show that our laminations are $C^{1+\alpha}$ in the strongest sense. In [8] we show how to use the stable and unstable ratio functions to construct uniformly bounded $C^{1+\alpha}$ orthogonal charts in which the images of the stable and unstable manifolds are respectively horizontal and vertical lines.

2. Statement of results

Throughout the paper, $f : M \rightarrow M$ is a $C^{1+\gamma}$ diffeomorphism of a compact manifold M , and Λ is a topologically transitive hyperbolic f -invariant set with local product structure [13]. If ρ is a $C^{1+\gamma}$ Riemannian metric on M and d denotes the corresponding distance on M , then, for $x \in M$, we denote the local stable and unstable manifolds through x by

$$W^i(x, \varepsilon) = \{y \in M : d(f_i^{-n}x, f_i^{-n}y) \leq \varepsilon, \text{ for all } n \geq 0\}. \quad (2.1)$$

Here $i = s$ or u (s for stable, u for unstable—a notation that we use throughout) and f_i is f if $i = u$ and f^{-1} if $i = s$. These sets are respectively contained in the stable and unstable manifolds

$$W^i(x) = \bigcup_{n \geq 0} f_i^n(W^i(f_i^{-n}x, \varepsilon_0))$$

which are immersed submanifolds.

We define *full i -leaf segments* I (which in the 1-dimensional case can intuitively be thought of as consisting of that part of the leaf between two points) as follows:

$I \subset W^1(x)$ is a *full ι -leaf segment* if it is connected and either (a) for all $x \in I$ for some $\varepsilon > 0$, $W^1(x, \varepsilon) \subset I$, or (b) I is the closure of such a set. Thus full ι -leaf segments are submanifolds of M . An *ι -leaf segment* is the intersection with Λ of a full ι -leaf segment.

Since Λ is a closed hyperbolic invariant set for f , for $\varepsilon > 0$ sufficiently small, there is $\delta = \delta(\varepsilon) > 0$ such that for all points $z, w \in \Lambda$ with $d(w, z) < \delta$, $W^s(z, \varepsilon)$ and $W^u(w, \varepsilon)$ intersect transversally in a unique point which we denote by $[w, z]$; see [13]. Since we assume that the hyperbolic set has a local product structure, we have $[w, z] \in \Lambda$. Furthermore, the following properties are satisfied:

- (i) $[w, z]$ varies continuously with $w, z \in \Lambda$;
- (ii) the bracket map is continuous on a δ -uniform neighbourhood of the diagonal in $\Lambda \times \Lambda$; and
- (iii) whenever both sides are defined, $f[z, w] = [fz, fw]$.

Note that the bracket map does not really depend on δ , provided that δ is sufficiently small.

We define a *rectangle* to be a subset of Λ which is (i) closed under the bracket (that is, $x, y \in R \Rightarrow [x, y] \in R$), and (ii) proper (that is, R is the closure of its interior in Λ). Since it demands properness, this definition is more restrictive than the usual one [13]. If $x \in R$, the spanning ι -leaf segment $\ell^1(x, R)$ is the intersection with Λ of the smallest connected full ι -leaf segment I containing x such that if $\iota = u$ every point z in R can be written in the form $[z, w]$, and if $\iota = s$ every point z in R can be written in the form $[w, z]$ for some $w \in I$.

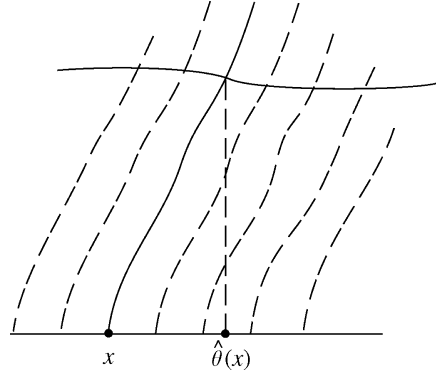
We now define the unstable basic holonomies; the stable basic holonomies are entirely analogous. Suppose that x and y are two points inside any rectangle R of Λ such that $y \in \ell^s(x, R)$. Let I and J be two unstable leaf segments respectively containing x and y , and inside R . Then we define $\theta : I \rightarrow \theta(I) \subset J$ by $\theta(w) = [y, w]$. Such maps are called the *basic u -holonomies*. They generate the pseudo-group \mathcal{H}^u of all unstable holonomies. Similarly, we define *basic s -holonomies* and \mathcal{H}^s .

THEOREM 2.1. *Consider a $C^{1+\gamma}$ diffeomorphism $f : M \rightarrow M$ with a codimension 1 hyperbolic invariant set Λ which is topologically transitive and has a local product structure. Suppose that the full ι -leaf segments are 1-dimensional. Then there is $0 < \alpha < 1$, such that all the basic ι -holonomies are $C^{1+\alpha}$.*

We shall also prove that these holonomies vary Hölder continuously with respect to the domain and target leaves. To state this result precisely, we need to introduce some new notions. For convenience, we henceforth restrict our attention to the unstable holonomies, and assume that the unstable leaves are 1-dimensional.

Fix a Riemannian metric ρ which is $C^{1+\gamma}$. Then, define $\mathcal{A}^u(\rho)$ to be the set of all maps $i : I \rightarrow \mathbb{R}$ where $I = \Lambda \cap \hat{I}$ with \hat{I} a full u -leaf segment such that if $W^u(x, \varepsilon) \subset \hat{I}$, then i extends to an isometry between the induced Riemannian metric on $W^u(x, \varepsilon)$ and the Euclidean metric on the reals. We call $\mathcal{A}^u(\rho)$ the *u -lamination atlas determined by ρ* . If I is a u -leaf segment (or a full u -leaf segment), then by $|I|_\rho$ we mean the length in the Riemannian metric ρ of the minimal full u -leaf containing I .

Consider a basic holonomy $\theta : I \rightarrow J$ between the u -leaf segments I and J . Suppose that the domains of $i, j \in \mathcal{A}^u(\rho)$ respectively contain I and J , and

FIGURE 1. The construction of the map $\hat{\theta}_y$.

suppose moreover that there is $x \in I$ such that $i(x) = j \circ \theta(x)$. Let $d_\Lambda(I, J)$ be as in Definition 3.2.

THEOREM 2.2. *There are $0 < \alpha, \beta < 1$ such that for all θ as above there is a diffeomorphic extension $\tilde{\theta}$ of $j \circ \theta \circ i^{-1}$ to \mathbb{R} such that*

$$\|\tilde{\theta} - \text{id}\|_{C^{1+\alpha}} \leq \mathcal{O}((d_\Lambda(I, J))^\beta), \quad (2.2)$$

where the constant of proportionality in the \mathcal{O} term depends only upon the choice of i , j and the rectangle R .

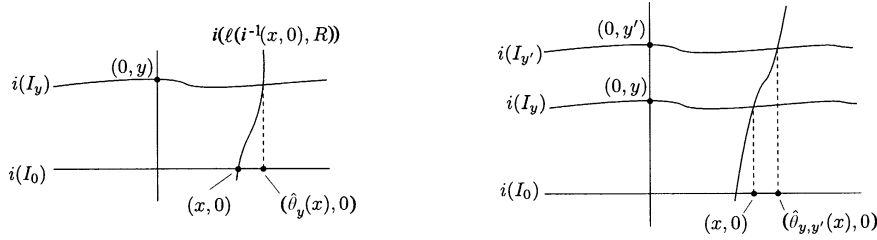
We now consider another natural way to express this continuity.

A chart $i : U \rightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ in the smooth structure on M is called *graph-like* if each full u -leaf segment and each local stable manifold of the form $W^s(x, \varepsilon)$ in U is respectively the graph of a C^{1+} function over the x -axis, and over the y -axis. Given such a chart and $x \in U$, by changing the coordinates by a local diffeomorphism of the form $(x, y) \rightarrow (x - u(y), y - v(x))$, we obtain a new chart $j : U \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}$ for which the images of the stable and unstable leaves through x are respectively contained in the y and x axes. We call such charts *straightened graph-like charts*. Clearly, one can choose an atlas of the smooth structure on M consisting of straightened graph-like charts.

Consider a straightened graph-like chart $i : U \rightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ and a rectangle R contained in U and containing $i^{-1}(0, 0)$. For $y \in \mathbb{R}$ with $(0, y)$ in the image of R under i , let $I_y = \ell(i^{-1}(0, y), R)$. Let $\pi : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be the projection into the first coordinate. Then we observe the following lemma.

LEMMA 2.3. *There is $0 < \alpha < 1$ such that if $j : I_y \rightarrow \mathbb{R}$ is in $\mathcal{A}^u(\rho)$, then $\pi \circ i \circ j^{-1}$ has a $C^{1+\alpha}$ diffeomorphic extension to \mathbb{R} . The $C^{1+\alpha}$ norm of the extension is bounded above by a quantity which depends only upon i , R and ρ .*

Consider the basic holonomies $\theta_y : I_0 \rightarrow I_y$ in R , and let $\hat{\theta}_y : \pi \circ i(I_0) \subset \mathbb{R} \rightarrow \mathbb{R}$ be given by $\hat{\theta}_y(x) = \pi \circ i \circ \theta_y \circ i^{-1}(x, 0)$; see Figure 1.

FIGURE 2. The maps $\hat{\theta}_y$ and $\hat{\theta}_{y,y'}$.

COROLLARY 2.4. *There are $0 < \alpha, \beta < 1$ such that the maps $\hat{\theta}_y$ have $C^{1+\alpha}$ diffeomorphic extensions $\tilde{\theta}_y$ to \mathbb{R} , and these can be chosen so that*

$$\|\tilde{\theta}_y - \tilde{\theta}_{y'}\|_{C^{1+\alpha}} \leq \mathcal{O}(|y - y'|^\beta), \quad (2.3)$$

where the constant of proportionality in the \mathcal{O} term depends only upon the choice of i and upon the rectangle R .

Consider the basic holonomy $\theta_{y,y'} : I_y \rightarrow I_{y'}$ in R , and let $\hat{\theta}_{y,y'} : \pi \circ i(I_y) \subset \mathbb{R} \rightarrow \mathbb{R}$ be given by $\hat{\theta}_{y,y'}(x) = \pi \circ i \circ \theta_{y,y'} \circ i^{-1}(x, 0)$; see Figure 2.

Then condition (2.3) is equivalent to the condition that for all such y and y' ,

$$\|\tilde{\theta}_{y,y'} - \text{id}\|_{C^{1+\alpha}} \leq \mathcal{O}(|y - y'|^\beta).$$

2.1. Notational conventions.

We use the notation $\phi = \mathcal{O}(\psi(x))$ to indicate that for all x , $|\phi(x)| < c|\psi(x)|$ where $c > 0$ is a constant which depends only upon quantities that are explicitly mentioned. Thus $\phi(n) = \mathcal{O}(v^n)$ means that $|\phi(n)| < cv^n$ for some constant c as above. Similarly, we use $\phi = \mathcal{O}(\psi(x))$ to indicate, for all x , that $c_1|\psi(x)| < |\phi(x)| < c_2|\psi(x)|$, where c_1 and c_2 are constants which depend only upon quantities that are explicitly mentioned. The notation $\phi(n) \in 1 \pm \mathcal{O}(v^n)$ means that there exists a constant $c > 0$, depending only upon explicitly mentioned quantities, such that $1 - cv^n < \phi(n) < 1 + cv^n$ for all $n \geq 0$.

3. Preliminary remarks

DEFINITION 3.1. A Markov partition of f is a collection $\mathcal{R} = \{R_1, \dots, R_m\}$ of rectangles such that

- (i) $\Lambda \subset \bigcup_{i=1}^m R_i$;
- (ii) $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all i and j ;
- (iii) if $x \in \text{int } R_i$ and $fx \in \text{int } R_j$ then
 - (a) $f\ell^s(x, R_i) \subset \ell^s(fx, R_j)$ and $f^{-1}\ell^u(fx, R_j) \subset \ell^u(x, R_i)$;
 - (b) $f\ell^u(x, R_i) \cap R_j = \ell^u(fx, R_j)$ and $f^{-1}\ell^s(fx, R_j) \cap R_i = \ell^s(x, R_i)$.

The last condition means that fR_i goes across R_j just once. In fact, it follows from condition (a), provided that the rectangles R_j are chosen sufficiently small [6]. The rectangles which make up the Markov partition are called *Markov rectangles*.

An ι -leaf primary cylinder is a spanning ι -leaf segment of a Markov rectangle. An ι -leaf n -cylinder is an ι -leaf segment I such that $f^n I$ is an ι -leaf primary cylinder

where $f_u = f$ and $f_s = f^{-1}$. For $n > 1$, an ι -leaf n -gap is a pair of distinct points x, y such that

- (i) for some rectangle R containing x and some full ι -leaf segment I ,
 $\{x, y\} = I \cap \Lambda$; and
- (ii) for $j = 0, \dots, n-1$, $f^j x$ lies in a Markov rectangle R_{i_j} and $f^j y \in \ell(f^j x, R_{i_j})$
 but this is not the case for $j = n$.

A primary ι -leaf gap is the image under f of an ι -leaf 1-gap.

We say that a rectangle R is an (n_s, n_u) -rectangle if there is $x \in R$ such that, for $\iota = s$ and u , each spanning leaf segment $\ell^\iota(x, R)$ is either an ι -leaf n_ι -cylinder or the union of two such cylinders with a common endpoint.

DEFINITION 3.2. If $x, y \in \Lambda$ and $x \neq y$, then $d_\Lambda(x, y) = 2^{-n}$, where n is the greatest integer such that both x and y are contained in an (n_s, n_u) -rectangle with $n_s, n_u \leq n$. Similarly, if I and J are ι -leaf segments, then $d_\Lambda(I, J) = 2^{-n'}$ where n' is the greatest integer such that both I and J are contained in an (n_s, n_u) -rectangle where $\iota' \in \{s, u\}$ is not ι .

4. Proofs of Theorems 2.1 and 2.2

Before proceeding to the proofs of the above theorems, we present a version of the naïve distortion lemma that we shall use. We shall consider the case where f is $C^{1+\gamma}$ and the full u -leaf segments are 1-dimensional. The case where the full s -leaf segments are 1-dimensional is analogous.

LEMMA 4.1. Let ρ be a $C^{1+\gamma}$ Riemannian metric as described above. Then for all u -leaf segments I and J with a common endpoint, and for all $n \geq 0$, we have

$$\left| \log \frac{|f^{-n}I|_\rho |J|_\rho}{|f^{-n}J|_\rho |I|_\rho} \right| \leq \mathcal{O}(|I \cup J|_\rho^\gamma), \quad (4.1)$$

where the constant of proportionality in the \mathcal{O} term depends only upon the choice of the Riemannian metric ρ .

Proof. Let \hat{I} and \hat{J} be the minimal full u -leaf segments such that $I = \hat{I} \cap \Lambda$ and $J = \hat{J} \cap \Lambda$. Also, let $k_n : f^{-n}(\hat{I} \cup \hat{J}) \rightarrow \mathbb{R}$ be an isometry between the Riemannian metric on the full u -leaf segments and the Euclidean metric on the reals.

The maps $\hat{f}_n : k_n \circ f^{-n}(\hat{I} \cup \hat{J}) \rightarrow k_{n+1} \circ f^{-(n+1)}(\hat{I} \cup \hat{J})$ defined by $\hat{f}_n = k_{n+1} \circ f^{-1} \circ k_n$ are $C^{1+\gamma}$ and have $C^{1+\gamma}$ norm uniformly bounded for all $n \geq 0$. Hence, by the mean value theorem and by the hyperbolicity of Λ for f , we get

$$\begin{aligned} \left| \frac{|f^{-n}I|_\rho |J|_\rho}{|f^{-n}J|_\rho |I|_\rho} \right| &\leq \sum_{i=0}^{n-1} \left| \log \hat{f}'_i(x_i) - \log \hat{f}'_i(y_i) \right| \\ &\leq \mathcal{O}(|I \cup J|^\gamma), \end{aligned}$$

where $x_i \in k_i \circ f^{-i}\hat{I}$ and $y_i \in k_i \circ f^{-i}\hat{J}$. □

We also need the following geometrical result.

LEMMA 4.2. *The lamination atlas $\mathcal{A}^u(\rho)$ has bounded geometry in the sense that*

- (i) *for all pairs I_1, I_2 of u -leaf n -cylinders or u -leaf n -gaps with a common point, we have $|I_1|_\rho/|I_2|_\rho$ uniformly bounded away from 0 and ∞ , with the bounds being independent of i, I_1, I_2 and n ;*
- (ii) *for all endpoints x and y of a u -leaf n -cylinder or u -leaf n -gap I , we have $|I|_\rho \leq \mathcal{O}((d_\Lambda(x, y))^\beta)$ and $d_\Lambda(x, y) \leq \mathcal{O}(|I|_\rho^\beta)$, for some $0 < \beta < 1$ which is independent of i, I and n .*

Proof. By the continuity of the stable and unstable bundles (see [3, Section 6]), the length $|I|_\rho$ of the leaf segments varies continuously with the endpoints. Thus by the compactness of Λ , the results follow for all pairs $I_1, I_2 \subset I_\theta$ of u -leaf 1-cylinders or u -leaf 1-gaps with a common point. Hence, by Lemma 4.1, we obtain the result for all pairs $I_1, I_2 \subset I_\theta$ of u -leaf n -cylinders or u -leaf n -gaps with a common point and for all $n > 1$. \square

Finally, the proofs of Theorems 2.1 and 2.2, and also of Corollary 2.4, use the following result, which follows directly from [7, Theorem 3].

THEOREM 4.3. *Suppose that $\theta : I \rightarrow J$ is a basic unstable holonomy for the rectangle R , and $i : I \rightarrow \mathbb{R}$ and $j : J \rightarrow \mathbb{R}$ are in $\mathcal{A}^u(\rho)$. The holonomy $\theta : I \rightarrow J$ is $C^{(1+\alpha)-}$ with respect to the charts of the lamination atlas $\mathcal{A}^u(\rho)$ for some $0 < \alpha < 1$, if and only if, for all $0 < \beta < \alpha$ and for all I_1 and I_2 with I_1 a leaf n -cylinder and I_2 a leaf n -cylinder or a leaf n -gap with a common endpoint with I_1 , we have*

$$\left| \log \frac{|j\theta I_1|}{|j\theta I_2|} \cdot \frac{|iI_2|}{|iI_1|} \right| \leq \mathcal{O}(|iK|^\beta)$$

whenever K is an unstable leaf segment containing I_1 and I_2 . Moreover, there are some $0 < \beta, \eta < 1$ and some affine map $a : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|j \circ \theta \circ i^{-1} - a\|_{C^{1+\eta}} \leq \mathcal{O}((d_\Lambda(I, J))^\beta) \quad (4.2)$$

if and only if there are some $0 < \beta, v < 1$ such that for all I_1 and I_2 as above we have

$$\left| \log \frac{|j\theta I_1|}{|j\theta I_2|} \cdot \frac{|iI_2|}{|iI_1|} \right| \leq \mathcal{O}((d_\Lambda(I, J))^\beta v^n).$$

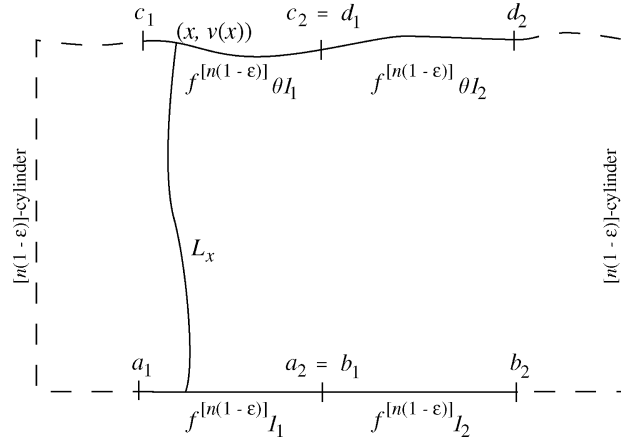
By $|K|$ we mean the Euclidean length of the minimal interval containing $K \subset \mathbb{R}$. For each of the \mathcal{O} terms, the constant of proportionality depends only upon the rectangle R .

Proof of Theorems 2.1 and 2.2. Fix a $C^{1+\gamma}$ Riemannian metric ρ and a finite atlas \mathcal{G} for M consisting of graph-like charts. If I is a leaf-segment, then by $|I|$ we mean the length $|I|_\rho$ in the Riemannian metric as defined above.

Let $I_1, I_2 \subset I_\theta$ be u -leaf n -cylinders or u -leaf n -gaps with a common point and $I = I_1 \cup I_2$. By Lemma 4.2, there are $0 < \psi \leq \alpha < 1$ such that for $0 \leq i \leq n$,

$$\mathcal{O}(\psi^{n-i}) \leq |f^i I_1|, |f^i I_2| \leq \mathcal{O}(\alpha^{n-i}). \quad (4.3)$$

Therefore, $|f^i I| \leq \mathcal{O}(\alpha^{n-i})$.

FIGURE 3. The various u leaf-segments in R' .

Let $[x]$ denote the integer part of $x \in \mathbb{R}$, and let $0 < \varepsilon < 1$. By Lemma 4.1, we have

$$\left| \log \frac{|I_1|}{|I_2|} \frac{|f^{[n(1-\varepsilon)]}I_2|}{|f^{[n(1-\varepsilon)]}I_1|} \right| \leq \mathcal{O} \left(|f^{[n(1-\varepsilon)]}I|^\gamma \right) \leq \mathcal{O}(\alpha^{\varepsilon\gamma n}). \quad (4.4)$$

Inequality (4.5) is also satisfied if we replace the leaf segment I_j by the leaf segment θI_j . Thus

$$\frac{|I_1|}{|I_2|} \cdot \frac{|\theta I_2|}{|\theta I_1|} \in (1 \pm \mathcal{O}(\alpha^{\varepsilon\gamma n})) \frac{|f^{[n(1-\varepsilon)]}I_1|}{|f^{[n(1-\varepsilon)]}I_2|} \cdot \frac{|f^{[n(1-\varepsilon)]}\theta I_2|}{|f^{[n(1-\varepsilon)]}\theta I_1|}. \quad (4.5)$$

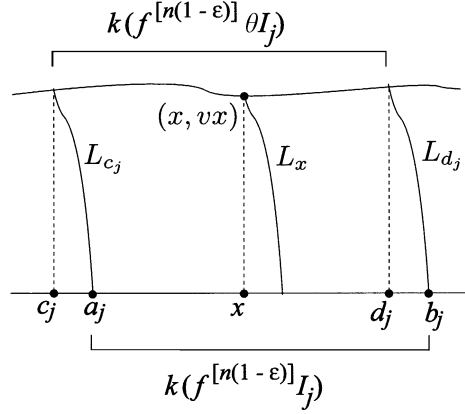
For $j = 1$ and 2 , $f^{[n(1-\varepsilon)]}I_j$ and $f^{[n(1-\varepsilon)]}\theta I_j$ are $[\varepsilon n]$ -cylinders contained in a rectangle R' whose spanning s leaf segments are contained in either an $[n(1-\varepsilon)]$ -cylinder or the union of two of them with a common endpoint.

Let us consider a graph-like chart $i : U \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}$ whose domain contains the rectangle R' . Let $u : (a, b) \rightarrow \mathbb{R}^{n-1}$ be the map whose graph contains the image under i of the full unstable leaf segment containing $f^{[n(1-\varepsilon)]}I_j$, and let $(a_j, u(a_j))$ and $(b_j, u(b_j))$ be the images under i of the endpoints of $f^{[n(1-\varepsilon)]}I_j$. By changing the coordinates by a local diffeomorphism of the form $(x, y) \rightarrow (x, y - u(x))$, we obtain a partially straightened graph-like chart $k : U \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}$ for which the image of $f^{[n(1-\varepsilon)]}I_j$ under k is contained in the horizontal axes. Let $v : (c, d) \rightarrow \mathbb{R}^{n-1}$ be the map for which the graph is the image under k of the stable or unstable manifold containing $f^{[n(1-\varepsilon)]}\theta I_j$, and let $(c_j, v(c_j))$ and $(d_j, v(d_j))$ be the images under i of the endpoints of $f^{[n(1-\varepsilon)]}\theta I_j$. If in this chart the Riemannian metric is given by $ds^2 = g_{11}dx^2 + 2g_{12}xdy + g_{11}dy^2$, then

$$\begin{aligned} |f^{[n(1-\varepsilon)]}I_j| &= \int_{a_j}^{b_j} (g_{11}(x, 0))^{1/2} dx, \\ |f^{[n(1-\varepsilon)]}\theta I_j| &= \int_{c_j}^{d_j} (g_{11}(x, v(x)) + 2g_{12}(x, v(x))v'(x) + g_{22}(x, v(x))v'(x)^2)^{1/2} dx. \end{aligned}$$

By $C^{1+\gamma}$ smoothness of the Riemannian metric, we obtain

$$|g_{11}(x, 0) - g_{11}(x, v(x))| \leq \mathcal{O}(|v(x)|^\gamma).$$

FIGURE 4. The leaves $f^{[n(1-\varepsilon)]}I$ and $f^{[n(1-\varepsilon)]}\theta I$.

By the Hölder continuity of the stable and unstable bundles (see [3, Section 6]), there is $0 < \eta < \gamma$ such that $|v'(x)| \leq \mathcal{O}(|v(x)|^\eta)$. Let L_x be the 1-dimensional submanifold with endpoints contained in the leaf segments $f^{[n(1-\varepsilon)]}I$ and $f^{[n(1-\varepsilon)]}\theta I$, such that the image under k of one of its endpoints is $(x, v(x))$, and such that L_x is contained in a full s -leaf segment (see Figure 4).

By hyperbolicity of Λ for f , there is $0 < \lambda < 1$ such that

$$\begin{aligned} |a_j - c_j| &\leq |(a_j, 0) - (c_j, v(c_j))| \leq \mathcal{O}(|L_{c_j}|) \leq \mathcal{O}(\lambda^{n(1-\varepsilon)}), \\ |b_j - d_j| &\leq |(b_j, 0) - (d_j, v(d_j))| \leq \mathcal{O}(|L_{d_j}|) \leq \mathcal{O}(\lambda^{n(1-\varepsilon)}), \quad \text{and} \\ |v(x)|^\eta &\leq \mathcal{O}(\lambda^{n\eta(1-\varepsilon)}). \end{aligned} \quad (4.6)$$

Thus, for $j \in \{1, 2\}$ and taking $\omega = \lambda^\eta < 1$, we have

$$||f^{[n(1-\varepsilon)]}\theta I_j| - |f^{[n(1-\varepsilon)]}I_j|| \leq \mathcal{O}(\omega^{n(1-\varepsilon)}). \quad (4.7)$$

Let $v \geq 0$ be such that $\omega^v = \psi$. By inequality (4.4), $|f^{[n(1-\varepsilon)]}I_j| \geq \mathcal{O}(\omega^{nv\varepsilon})$. Therefore,

$$\left| \log \frac{|f^{[n(1-\varepsilon)]}\theta I_j|}{|f^{[n(1-\varepsilon)]}I_j|} \right| \leq \mathcal{O}(\omega^{n(1-\varepsilon)(1+v)}). \quad (4.8)$$

Choose $0 < \varepsilon < 1$ such that $0 < \mu = \max\{\alpha^{\varepsilon\gamma}, \omega^{1-\varepsilon(1+v)}\} < 1$. By inequality (4.6) and inequality (4.9), we obtain

$$\left| \log \frac{|I_1|}{|I_2|} \frac{|\theta I_2|}{|\theta I_1|} \right| \leq \mathcal{O}(\mu^n). \quad (4.9)$$

Since this is true for all $n > 0$, and for every I_1 which is a u -leaf n -cylinder and every I_2 which is either a u -leaf n -cylinder or a u -leaf n -gap and has one common endpoint with I_1 , it follows by Theorem 4.3 that the holonomy $\theta : I_\theta \rightarrow J_\theta$ is $C^{1+\beta}$ for some $\beta = \beta(\mu) > 0$ where β depends only upon μ . This proves Theorem 2.1.

Now we prove Theorem 2.2; that is, we prove that the holonomy $\theta : I_\theta \rightarrow J_\theta$ varies Hölder continuously with respect to I_θ, J_θ . As in our proof of inequality (4.8), we deduce that there is $0 < \varepsilon_1 < 1$ such that $||I_j| - |\theta I_j|| \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_1})$ for $j \in \{1, 2\}$. Now, we choose η small enough, so that $0 < \rho = \eta^{\varepsilon_1/2} \psi^{-1} < 1$. If $d_\Lambda(I_\theta, J_\theta) \leq \mathcal{O}(\eta^n)$ then, as in inequality (4.9),

$$\left| \log \frac{|\theta I_j|}{|I_j|} \right| \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_1/2} \eta^{n\varepsilon_1/2} \psi^{-n}) \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_1/2} \rho^n). \quad (4.10)$$

Therefore,

$$\left| \log \frac{|I_1|}{|\theta I_1|} \frac{|\theta I_2|}{|I_2|} \right| \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_1/2} \rho^n).$$

Let $\varepsilon_2 > 0$ be such that $\mu = \eta^{2\varepsilon_2}$. If $d_\Lambda(I_\theta, J_\theta) \geq \mathcal{O}(\eta^n)$ then, by inequality (4.10),

$$\left| \log \frac{|I_1|}{|I_2|} \frac{|\theta I_2|}{|\theta I_1|} \right| \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_2} \mu^{n/2}).$$

Therefore, by Theorem 4.3, there is an affine map $a : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|j \circ \theta \circ i^{-1} - a\|_{C^{1+\alpha}} \leq \mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_2}).$$

By inequality (4.11) and since there is a point x such that $j \circ \theta \circ i^{-1}(x) = x$, we get from the last inequality that a is $\mathcal{O}((d_\Lambda(I_\theta, J_\theta))^{\varepsilon_3})$ -close to the identity in the $C^{1+\alpha}$ -norm for some $\varepsilon_3 > 0$, and so inequality (2.2) follows. \square

Proof of Lemma 2.3. Let $I_1, I_2 \subset I_y$ be u -leaf n -cylinders or u -leaf n -gaps with a common point and $I = I_1 \cup I_2$. Let $I_\pi = \pi \circ i(I)$ and let $I_{\pi,k} = \pi \circ i(I_k)$ for k equal to 1 and 2. Since $|I_\pi| = \mathcal{O}(|I|)$, we obtain by (4.4) that there are $0 < \psi \leq \alpha < 1$ such that

$$\mathcal{O}(\psi^n) \leq |I_\pi| \leq \mathcal{O}(\alpha^n). \quad (4.11)$$

The image of the full u -leaf segment \hat{I}_y with $\hat{I}_y \cap \Lambda = I_y$ under i is a graph of the form $(x, v_y(x))$ where v_y is $C^{1+\gamma}$. Letting a_k and b_k be the image under i_y of the endpoints of I_k , we find that

$$|I_k| = \int_{a_k}^{b_k} (g_{11}(x, v(x)) + 2g_{12}(x, v(x))v'(x) + g_{22}(x, v(x))v'(x)^2)^{1/2} dx.$$

Since v_y is $C^{1+\gamma}$, we obtain

$$|v'_y(w) - v'_y(z)| \leq \mathcal{O}(|I_\pi|^\gamma) \leq \mathcal{O}(\alpha^{n\gamma}) \quad (4.12)$$

for all $w, z \in I_\pi$. By the Hölder continuity of the Riemannian metric, there is $0 < \eta \leq 1$ such that

$$|g_{j,l}(w) - g_{j,l}(z)| \leq \mathcal{O}(|I_\pi|^\eta) \leq \mathcal{O}(\alpha^{n\eta}) \quad (4.13)$$

for all $w, z \in I_\pi$. Let $v = \max\{\alpha^\gamma, \alpha^\eta\}$. By (4.13) and (4.14), and taking t such that $|I_1| = t |I_{\pi,1}|$, we obtain

$$|I_2| = t |I_{\pi,2}| (1 \pm \mathcal{O}(v^n)).$$

Hence

$$\left| \log \frac{|I_2|}{|I_1|} \frac{|I_{\pi,1}|}{|I_{\pi,2}|} \right| \leq \mathcal{O}(v^n), \quad (4.14)$$

and so by Theorem 4.3 the overlap map $\pi \circ i \circ j^{-1}$ has a $C^{1+\alpha}$ diffeomorphic extension to \mathbb{R} with $C^{1+\alpha}$ norm bounded above by a quantity which depends only upon i, R and ρ . \square

Proof of Corollary 2.4. Let $I_1, I_2 \subset I_y$ be u -leaf n -cylinders or u -leaf n -gaps with a common point and $I = I_1 \cup I_2$. Let $I_\pi = \pi \circ i(I)$, $J = h(I)$ and $J_\pi = \pi \circ i(J)$. For $k \in \{1, 2\}$, let $I_{\pi,k} = \pi \circ i(I_k)$, $J_k = h(I_k)$ and $J_{\pi,k} = \pi \circ i(J_k)$. By (4.10), there is $0 < v < 1$ such that

$$\left| \log \frac{|I_1|}{|I_2|} \frac{|J_2|}{|J_1|} \right| \leq \mathcal{O}(v^n).$$

Thus, by (4.15) we obtain

$$\left| \log \frac{|I_{\pi,1}|}{|I_{\pi,2}|} \frac{|J_{\pi,2}|}{|J_{\pi,1}|} \right| \leq \mathcal{O}(v^n). \quad (4.15)$$

Therefore, by Theorem 4.3, the map $\hat{\theta}_{y,y'}$ has a $C^{1+\alpha}$ diffeomorphic extension to \mathbb{R} .

Let L_z be the 1-dimensional submanifold contained in a full s -leaf segment with minimal length and with endpoints $z \in I_y$ and $\theta_{y,y'}(z) \in I_{y'}$. By the hyperbolicity of Λ for f , there is $0 < \varepsilon_1 \leq 1$ such that

$$|\pi \circ i(x) - \pi \circ i \circ \theta_{y,y'}(x)| \leq \mathcal{O}(|L_z|) \leq \mathcal{O}(|y - y'|^{\varepsilon_1}).$$

Thus, for $k \in \{1, 2\}$

$$||J_{\pi,k}| - |I_{\pi,k}|| \leq \mathcal{O}(|y - y'|^{\varepsilon_1}). \quad (4.16)$$

Let ψ be as in (4.12). Choose η small enough such that $0 < \tau = \eta^{\varepsilon_1/2} \psi^{-1} < 1$. If $|y - y'| \leq \mathcal{O}(\eta^n)$, then by (4.12) and (4.17) we obtain

$$\left| \log \frac{|J_{\pi,k}|}{|I_{\pi,k}|} \right| \leq \mathcal{O}(|y - y'|^{\varepsilon_1/2} \eta^{n\varepsilon_1/2} \psi^{-n}) \leq \mathcal{O}(|y - y'|^{\varepsilon_1/2} \tau^n). \quad (4.17)$$

Therefore,

$$\left| \log \frac{|J_{\pi,1}|}{|I_{\pi,1}|} \frac{|I_{\pi,2}|}{|J_{\pi,2}|} \right| \leq \mathcal{O}(|y - y'|^{\varepsilon_1/2} \tau^n).$$

Let $\varepsilon_2 > 0$ be such that $v = \eta^{2\varepsilon_2}$. If $|y - y'| \geq \mathcal{O}(\eta^n)$, then by inequality (4.16)

$$\left| \log \frac{|J_{\pi,1}|}{|J_{\pi,2}|} \frac{|I_{\pi,2}|}{|I_{\pi,1}|} \right| \leq \mathcal{O}(|y - y'|^{\varepsilon_2} v^{n/2}).$$

Therefore, by Theorem 4.3 there is an affine map $a : \mathbb{R} \rightarrow \mathbb{R}$ and $0 < \alpha \leq 1$ such that

$$||\tilde{\theta}_{y,y'} - a||_{C^{1+\alpha}} \leq \mathcal{O}(|y - y'|^{\varepsilon_2}).$$

Since $\hat{\theta}_{y,y'}(0) = 0$ and by (4.18), there is $\varepsilon_3 > 0$ such that a is $\mathcal{O}(|y - y'|^{\varepsilon_3})$ close to the identity in the $C^{1+\alpha}$ norm, and so (2.3) follows. \square

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