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Explosion of smoothness from a point to everywhere for conjugacies between Markov families

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Abstract. For uniformly asymptotically affine (uaa) Markov maps on train tracks, we prove the following type of rigidity result: if a topological conjugacy between them is (uaa) at a point in the train track then the conjugacy is (uaa) everywhere. In particular, our methods apply to the case in which the domains of the Markov maps are Cantor sets. We also present similar statements for (uaa) and C^r Markov families. These results generalize the similar ones of Sullivan and de Faria for C^r expanding circle maps with $r > 1$ and have useful applications to hyperbolic dynamics on surfaces and laminations.

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1. Introduction

Sullivan (1991) stated the following theorem: if a topological conjugacy between two C^r expanding circle maps with $r > 1$ is smooth at a point then the conjugacy is smooth everywhere. de Faria (1996) proved a more general result, showing that if the conjugacy between two C^r expanding circle maps is uniformly asymptotically affine (uaa) at a point then the conjugacy is smooth everywhere. Here, we prove that a topological conjugacy between two (uaa) Markov maps which is (uaa) at a point is (uaa) everywhere (see a more general statement in Theorem 1). Furthermore, if the Markov maps are C^r and the conjugacy is (uaa) we show that the conjugacy is C^r. These results are also satisfied for Markov families, as we prove in section 3. We observe that throughout this paper we say that a map ϕ is C^r if its kth derivative is α-Hölder continuous, where $k \in \mathbf{N}$, $0 < \alpha \leq 1$ and $r = k + \alpha$ (in particular, if r is an integer the map is $r - 1$ times differentiable with the Lipschitz $(r-1)$th derivative).

Our results generalize the previous ones in two ways. Firstly, instead of expanding circle maps we consider Markov maps on train tracks for which the domains can be Cantori. As we will explain below, this generalization is essential to extend in Ferreira and Pinto (2001) these results to another much studied class of maps consisting of diffeomorphisms on surfaces with hyperbolic basic sets. Secondly, we

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consider (uaa) Markov maps that form a more general class than the C^r Markov maps. This regularity appears in this context as a natural limit of the $C^{1+\alpha}$ regularity (when α tends to 0) instead of the usual C^1. In section 1.1, we point out several properties of the (uaa) regularity which make this regularity useful and important in dynamics. Our results are natural in this regularity having conceptual and geometric proofs. Then, we easily extend them to the C^r regularity using canonical charts (see section 2.6).

An interesting feature of the theorems proved in this paper is that they show an unexpected rigidity property for the conjugacy between these systems since, in general, the conjugacies between Markov maps are just Hölder continuous but under the weak assumption of the conjugacy being (uaa) at a point we show that the conjugacy is (uaa) everywhere, or even C^r if the Markov maps are C^r. From a practical point of view these results are useful since, for instance, it is much easier to check that a map is (uaa) or C^1 at a point than everywhere.

Markov maps with smooth structures on train tracks as presented here were first introduced in Pinto and Rand (1995). Besides forming an interesting mathematical object on their own they are also essential to understand hyperbolic dynamics on surfaces. For instance, in Pinto and Rand (2001) they are used to give a full classification of smooth conjugacy classes of hyperbolic dynamics on surfaces and to construct geometric measures for these systems. In Ferreira and Pinto (2001) they are used together with the results of this paper to prove that if a topological conjugacy between two diffeomorphisms on surfaces with hyperbolic basic sets is smooth at a point in the hyperbolic basic set then the conjugacy has a smooth extension to the surface. This also shows that the results of this paper are already relevant in the C^r context.

1.1. *Uniformly asymptotically affine regularity*

Sullivan (1993) in a fundamental paper on convergence under renormalization of infinitely renormalizable maps introduced the notion of (uaa) expanding circle maps. There Sullivan proves that the (uaa) conjugacy classes of expanding circle maps form the natural completion of the $C^{1+\alpha}$ conjugacy classes, where $\alpha > 0$. In Ferreira (1994) and here we generalize the notion of (uaa) maps to the case where the domains can be Cantori. In Remark 1, we prove that our definition coincides with the classical one in the case where the domains are intervals.

In Ferreira (1994) we also extend the result of Sullivan to (uaa) Markov maps on train tracks showing that the (uaa) conjugacy classes form a natural completion of the $C^{1+\alpha}$ conjugacy classes. In particular, we prove that (uaa) conjugacy classes of 'cookie-cutters' (see definition in section 2.2.1) are in one-to-one correspondence with continuous functions $s : \{0,1\}^{\mathbf{N}} \to \mathbf{R}^+$ (which we call the 'solenoid functions'). Furthermore, we show that if a (uaa) conjugacy class has a $C^{1+\alpha}$ map contained in it then the corresponding solenoid function is Hölder continuous. Since there are solenoid functions that are not Hölder continuous, we obtain the existence of (uaa) cookie-cutters that are not $C^{1+\alpha}$ and even more they are not (uaa) conjugate to a $C^{1+\alpha}$ one. For a given solenoid function (not Hölder continuous), we also construct in Ferreira (1994) a corresponding geometric realization of a (uaa) cookie-cutter.

(Uaa) maps are also relevant in other mathematical contexts. Start by noting that from the Beurling–Ahlfors extension theorem every quasisymmetric homeomorphism of $\hat{\mathbf{R}}$ can be extended to a quasiconformal homeomorphism of the upper half plane (we say that a homeomorphism h is quasisymmetric if the modulus of continuity χ_c of h in Definition 1 is just a bounded function). In Gardiner and Sullivan (1992), it is proved that (uaa) (or equivalently, symmetric) homeomorphisms are the boundary values of quasiconformal homeomorphisms of the upper half plane whose conformal distortion tends to zero at the boundary. (Uaa) homeomorphisms turn out to be precisely those homeomorphisms that have boundary dilatation equal to one, in the sense of Strebel (1976). The (uaa) homeomorphisms of a circle comprise the closure, in the quasisymmetric topology, of the real analytic homeomorphisms and this closure contains the set of C^1 diffeomorphisms. In Sullivan (1991), a one-to-one correspondence is shown between (uaa) conjugacy classes of expanding circle maps and complex structures on a solenoidal surface. Another application of (uaa) maps appears in the following extension of the classic Arnold–Herman–Yoccoz rigidity theorem for diffeomorphisms of the circle: a $C^{1+\mathrm{zigmund}}$ diffeomorphism of the circle with golden rotation number is (uaa) conjugate to the rigid golden rotation (see Ferreira *et al.* (2001)).

As we explain next, the definition of a (uaa) Markov map is a geometric notion consisting in a bound of the ratio distortion for triples of points under iteration. This bound is slightly weaker than the bound satisfied by smooth Markov maps. If a Markov map is $C^{1+\alpha}$ then the modulus of continuity $\chi(t)$ (controlling the ratio distortion) for compositions of inverse Markov maps satisfies the inequality $\chi(t) < \mathcal{O}(|t|^{\alpha})$, where $0 < \alpha < 1$ (see inequality (2)). The (uaa) regularity is characterized by demanding only that $\chi(t)$ converges to zero when t tends to zero. Hence, the (uaa) Markov maps arise as a natural limit on the degree $1 + \alpha$ of smoothness of the Markov maps when α tends to 0, instead of the usual C^1 smoothness.

Definition 1. The local homeomorphism $\phi : I \subset \mathbf{R} \to \mathbf{R}$ is 'uniformly asymptotically affine (uaa) at a point $x \in I$' if for all $c \geq 1$ there is a continuous function $\chi_c : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying $\chi_c(0) = 0$ such that for all points $y_1, y_2, y_3 \in I$ with $c^{-1} \leq (y_3 - y_2)/(y_2 - y_1) \leq c$, we have

$$\left|\log \frac{\phi(y_2) - \phi(y_1)}{\phi(y_3) - \phi(y_2)} \frac{y_3 - y_2}{y_2 - y_1}\right| < \chi_c(\max\{|y_3 - x|, |y_1 - x|\}). \tag{1}$$

We call χ_c the 'modulus of continuity' of ϕ. The left-hand side of (1) is called the 'ratio distortion' of ϕ at the points y_1, y_2 and y_3.

The local homeomorphism $\phi : I \to \mathbf{R}$ is (*uaa*) if ϕ is (uaa) at every point $x \in I$ with modulus of continuity χ_c not depending upon the point x.

We say that $\phi : I \to \mathbf{R}$ is 'asymptotically affine (aa) at a point $x \in I$' if ϕ satisfies inequality (1) in the case where $y_2 = x$.

The classical definition of a (uaa) or symmetric function ϕ is given by taking $c = 1$. Here, we consider in the definition all $c \geq 1$ because I does not have to be an interval. For instance I can be a Cantor set. However, by the following remark these two conditions are equivalent if I is an interval.

Remark 1. If I is an interval and if, for $c = 1$, ϕ satisfies inequality (1) for all $x \in I$ then ϕ satisfies that inequality for all $c > 1$.

The proof of Remark 1 is in section 4. Next, we present our result for (uaa) Markov maps with respect to atlases A and B (see definitions in section 2).

Theorem 1. *Let M and N be topologically conjugate* (*uaa*) *Markov maps with respect to atlases A and B, respectively. If the conjugacy is* (*aa*) *at the points of a periodic orbit or is* (*aa*) *at a point with dense orbit, then the conjugacy is* (*uaa*). *If the conjugacy is* (*uaa*) *at a point then the conjugacy is* (*uaa*). *Furthermore, if M and N are* C^r *with* $r > 1$ *then the* (*uaa*) *conjugacy is* C^r.

If the conjugacy is differentiable at a point then it is (uaa) at this point and so we obtain the following corollary (see the proof in section 3.5).

Corollary 1. *Let M and N be topologically conjugate* C^r *Markov maps with respect to atlases A and B, respectively. If the conjugacy is differentiable at a point then the conjugacy is* C^r.

In section 3, we prove a more general result than that presented in Theorem 1 which also applies to Markov families (see Theorems 2–5). The reason that we are interested in this generalization is that the Markov families are used in more general situations than Markov maps, such as the classification of smooth foliations on a torus and the classification of smooth conjugacy classes of infinitely renormalizable unimodal maps, circle maps, and interval exchange maps (see Pinto and Rand (1991), Ferreira *et al.* (2001)).

2. Markov families on train tracks

2.1. *Train tracks*

Let $\tilde{T} = \sqcup \tilde{C}_i/_\sim$ be the disjoint union of closed intervals $\tilde{C}_i$ of $\mathbf{R}$ with an equivalence relation $\sim$ on the endpoints of the intervals $\tilde{C}_i$. A set $\tilde{I} \subset \tilde{T}$ is an 'open segment of $\tilde{T}$' if, for every $x \in \tilde{I}$, $\mathrm{cl}(\tilde{I}) \setminus \{x\}$ has two connected components $\tilde{I}_1$ and $\tilde{I}_2$. A 'closed segment' $\tilde{J} \subset \tilde{T}$ is the closure $\mathrm{cl}(\tilde{I})$ of an open segment $\tilde{I}$. The boundary of an (open or closed) segment $\tilde{I}$ is $\partial\tilde{I} = \mathrm{cl}(\tilde{I}) \setminus \mathrm{int}\ \tilde{I}$.

We say that $\tilde{S}$ is an 'admissible set of open segments of $\tilde{T}$' if it satisfies the following properties: (i) if $I \in \tilde{S}$ then I is an open segment of $\tilde{T}$; (ii) for all $x \in \tilde{T}$ there exists $I \in \tilde{S}$ which contains x; and (iii) if I is an open segment of $\tilde{T}$ and I is contained in a union of segments in $\tilde{S}$ then I is also in $\tilde{S}$.

Let T be a (compact and proper) subset of $\tilde{T}$, and $\tilde{S}$ an admissible set of open segments of $\tilde{T}$. We say that Δ_{O} is an admissible set of open segments of T if there is an admissible set $\tilde{S}$ of open segments of $\tilde{T}$ such that $\Delta_{\mathrm{O}} = \{\tilde{I} \cap T : \tilde{I} \in \tilde{S}\}$. We say that J is a 'closed segment of T' if there is an open segment $I \subset \Delta_{\mathrm{O}}$ such that $J = \mathrm{cl}(I)$. Let Δ be the set of all open and closed segments of T determined by Δ_{O}. The 'boundary ∂I of a segment I' of T is the boundary of the smallest segment $\tilde{I} \subset \tilde{T}$ such that $I = \tilde{I} \cap T$. The 'interior int I of a segment I' of T is $\mathrm{int}\ I = I \setminus \partial I$. The triple $(T, \tilde{T}, \Delta)$ forms a 'train track' T_Δ.

Let $T_\Delta = (T, \tilde{T}, \Delta)$ be a train track. A 'chart' (i, I) is a map $i : I \to \mathbf{R}$ which is the restriction of an injective and continuous map $\tilde{i} : \tilde{I} \to \mathbf{R}$, where $\tilde{I}$ is an open segment of $\tilde{T}$ and $\tilde{I} \cap T = I \in \Delta_{\mathrm{O}}$. An 'atlas A on T_Δ' is a set of charts with the property that for every $x \in T$ and $J \in \Delta_{\mathrm{O}}$ with $x \in J$, there exists a chart (i, I) such that $I \cap J$ contains an open segment K with $x \in K$. We note that (for simplicity of exposition) if (i, I) is in A we will consider that $(i|I', I')$ is also in A for every interval $I' \subset I$. Two charts (i, I) and (j, J) with $I, J \subset T$ are '(uaa) compatible' if the overlap map $i \circ j^{-1} : j(I \cap J) \to i(I \cap J)$ is (uaa) when $I \cap J \neq \emptyset$. A '(uaa) atlas A on T_Δ' is an atlas formed by charts which are (uaa) compatible.

Let $T_\Delta = (T, \tilde{T}, \Delta)$ and $P_\Gamma = (P, \tilde{P}, \Gamma)$ be train tracks. The map $h : I \subset T \to J \subset P$ is a homeomorphism if there are connected sets $\tilde{I} \subset \tilde{T}$ and $\tilde{J} \subset \tilde{P}$ with $I = \tilde{I} \cap T$ and $h(I) = \tilde{J} \cap P$ such that h extends to a homeomorphism $\tilde{h} : \tilde{I} \to \tilde{J}$ and the image of every segment in $\tilde{I}$ is a segment in I, and vice versa. Let A and B be atlases on T_Δ and on P_Γ, respectively. The homeomorphism $h : I \subset T \to J \subset P$ is (aa) at $x \in T$ if for every chart $(i, I') \in A$ with $x \in I' \subset I$ and every chart $(j, J') \in B$ with $h(x) \in J' \subset J$ we have that $j \circ h \circ i^{-1} | i(I' \cap h^{-1}(J'))$ is (aa) at $i(x)$ with modulus of continuity not depending upon the charts considered. The homeomorphism $h : I \subset T \to J \subset P$ is (uaa) at $x \in T$ if for every chart $(i, I') \in A$ with $x \in I' \subset I$ and every chart $(j, J') \in B$ with $h(x) \in J' \subset J$ we have that $j \circ h \circ i^{-1} | i(I' \cap h^{-1}(J'))$ is (aa) at $i(x)$ with modulus of continuity not depending upon the charts considered. The homeomorphism h is (uaa) if h is (uaa) at every point $x \in I$ with modulus of continuity χ_c not depending upon the point x.

2.2. *Markov families*

For every $n \in \mathbf{Z}$, let $T^n_\Delta = (T^n, \tilde{T}^n, \Delta^n)$ be a train track and $M_n : T^n \to T^{n+1}$ a map. A 'Markov partition of $(M_n, T^n_\Delta)_{n \in \mathbf{Z}}$' is a collection $(C^n_1, \ldots, C^n_{m(n)})_{n \in \mathbf{Z}}$ of closed and proper segments in Δ^n with the following properties for every $n \in \mathbf{Z}$.

(i) $T^n = \bigcup_{i=1}^{m(n)} C^n_i$ and the constant $m(n)$ is bounded away from infinity independently of n.
(ii) $\operatorname{int} C^n_i \bigcap \operatorname{int} C^n_j = \varnothing$ if $i \neq j$.
(iii) $M_n | \operatorname{int} C^n_i$ is a homeomorphism on to its image.
(iv) If $x \in \operatorname{int} C^n_i$ and $M_n(x) \in C^{n+1}_j$ then $M_n(C^n_i)$ contains C^{n+1}_j.
(v) For every $C^{n+1}_j \subset T^{n+1}$, there exists a C^n_i such that $M_n(C^n_i)$ contains C^{n+1}_j.
(vi) Let

$$C^n_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_m} = \{x \in C^n_{\varepsilon_1} : (M_{n+j-1} \circ \cdots \circ M_n)(x) \in C^{n+j}_{\varepsilon_j}, j = 1, 2, \ldots, m-1\}$$

be an 'm-cylinder' if $C^n_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m} \neq \varnothing$. For every sequence $C^n_{\varepsilon_1}$, $C^n_{\varepsilon_1 \varepsilon_2}$, ... of cylinders, $\lim_{i \to \infty} \bigcap_{m=1}^{i} C^n_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_m}$ is a single point.
(vii) For every C^n_i, there exists $l = l(i, n)$ such that $T^{n+l} = M^l_n(C^n_i)$, where $l(i, n)$ is bounded away from infinity independently of i and n.
(viii) For every open segment K and $x \in K$, there is an open segment I such that $M_n(I) \subset K$ and $x \in M_n(I)$.

An 'm-gap' G^n is a closed segment contained in an $(m-1)$-cylinder with the property that G^n is equal to two points that are endpoints of two m-cylinders (in particular, G^n is equal to its boundary).

Definition 2. A 'Markov family $(M_n, T^n_\Delta)_{n \in \mathbf{Z}}$' is a sequence of train tracks $T^n_\Delta = (T^n, \tilde{T}^n, \Delta^n)$ and maps $M_n : T^n \to T^{n+1}$ with a Markov partition. A 'Markov map (M, T_Δ)' is a Markov family $(M_n, T^n_\Delta)_{n \in \mathbf{Z}}$, where $M_n = M$ and $T^n_\Delta = T_\Delta$ for every $n \in \mathbf{Z}$.

2.2.1. *Example 1: cookie-cutters.*

The following example of a Markov map occurs naturally in the study of Smale horseshoes (see Ferreira and Pinto (2001)), and in this case T is a Cantor set.

Let $\tilde{T} = [0, 1]$ and

$$\tilde{M}(x) = \begin{cases} 3x, & \text{for } x \in [0, 1/3], \\ 3x - 2, & \text{for } x \in [2/3, 1]. \end{cases}$$

Let $T = \bigcap_{n=0}^{\infty} \tilde{M}^{-n}([0,1])$. We choose the set of all admissible segments Δ in T to be the set of all open segments of $\tilde{T}$ intersected with T. Finally note that $C_1 = [0, 1/3] \cap T$ and $C_2 = [2/3, 1] \cap T$ form a Markov partition for the map $M = \tilde{M}|T$.

2.2.2. *Example 2: horocycle maps.* The following example of a Markov map occurs naturally in the study of Anosov maps (see Ferreira and Pinto (2001)), and in this case $T = \tilde{T}$ is not a one-dimensional branched manifold.

Let $\tilde{C}_1 = [-1, 0]$ and $\tilde{C}_2 = [0, \gamma]$ where $\gamma = (\sqrt{5} - 1)/2$ is the golden number. Let $\tilde{T} = \tilde{C}_1 \sqcup \tilde{C}_2 / \sim$ where all the endpoints of $\tilde{C}_1$ and $\tilde{C}_2$ are in the same equivalence class. Let the set $\tilde{S}$ of all admissible segments of $\tilde{T}$ be given as follows: for all $-1 < a < b < 0 < c < d < \gamma$,

(i) $(a, b) \subset \tilde{T}$;
(ii) $(c, d) \subset \tilde{T}$;
(iii) $(a, 0] \cup [0, d) \subset \tilde{T}$;
(iv) $(d, \gamma] \cup [-1, a) \subset \tilde{T}$;
(v) $(b, 0] \cup [-1, a) \subset \tilde{T}$;
(vi) $(b, 0] \cup [0, \gamma] \cup [-1, a) \subset \tilde{T}$ (see figure 1).

Let $T = \tilde{T}$ and $\Delta = \tilde{S}$. Let $M : T \to T$ be given by

$$M(x) = \begin{cases} -\gamma^{-1}x - 1, & \text{for } x \in [-1, 0], \\ -\gamma^{-1}x, & \text{for } x \in [0, \gamma]. \end{cases}$$

Then M admits a Markov partition.

2.3. (*Uaa*) *Markov families*

We start by presenting a definition that will make the notation simpler and clearer throughout the paper.

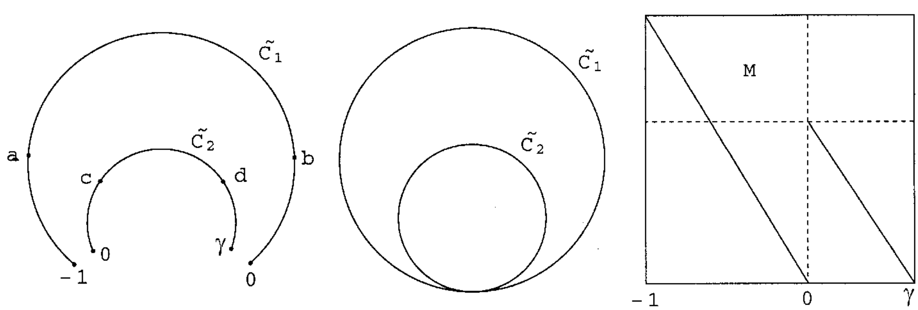

Figure 1. The representation of the train track and the Markov map for the Example 2.

Definition 3. Let $(M_n, T^n_\Delta)_{n\in\mathbf{Z}}$ be a Markov family, $(A_n)_{n\in\mathbf{Z}}$ a family of atlas A_n on T^n_Δ, and (i, I) a chart in the atlas A_n. For all distinct points $x, y, z \in I$ with $i(y)$ lying between $i(x)$ and $i(z)$, we define the 'ratio $r_i(x, y, z)$' by

$$r_i(x,y,z) = \frac{i(z) - i(y)}{i(y) - i(x)}.$$

For every segment $K \subset I$ we denote by $|K|_i$ the length of the smallest interval that contains $i(K)$. For simplicity of notation, we will use $r(x, y, z)$ and $|K|$ instead of $r_i(x, y, z)$ and $|K|_i$, respectively, when it is clear which is the chart that we are considering.

We note that the set of all ratios $r_i(x, y, z)$ determines the chart (i, I) up to affine composition.

Let $(M_n, T^n_\Delta)_{n\in\mathbf{Z}}$ be a Markov family and $(A_n)_{n\in\mathbf{Z}}$ a family of atlas A_n on T^n_Δ. Given two open segments $I \subset T^m$ and $J \subset T^n$ we denote by $M_{IJ} : I \to J$ the map $M_{IJ} = M_{n-1} \circ \cdots \circ M_m | I$ if M_{IJ} is a homeomorphism. We say that $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is a '(uaa) Markov family' if it satisfies the following properties.

(i) For every $c \geq 1$, there exists a continuous function $\chi_c : \mathbf{R}^+_0 \to \mathbf{R}^+_0$ with $\chi_c(0) = 0$ such that for all homeomorphisms $M_{IJ} : I \to J$, for all charts $(i, I) \in A_m$ and $(j, J) \in A_n$, and for all points $x, y, z \in J$ with $c^{-1} \leq r(x, y, z) \leq c$, we have

$$\left| \log \frac{r(M^{-1}_{IJ}(x), M^{-1}_{IJ}(y), M^{-1}_{IJ}(z))}{r(x,y,z)} \right| < \chi_c(|z - x|). \tag{2}$$

(ii) For every closed segment I which is a 1-cylinder or a union of two 1-cylinders with a common endpoint, there is a chart $(i, I') \in A_n$ such that $I \subset I'$. There exists a constant $b > 1$ such that for every 2-cylinder or 2-gap I, $b^{-1} < |I|_i < b$ for every chart (i, I') with $I \subset I'$.

We call χ_c the 'modulus of continuity of $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$'.

A '(uaa) Markov map' (M, T_Δ, A) is a (uaa) Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$, where $M_n = M$, $T^n_\Delta = T_\Delta$ and $A_n = A$ for every $n \in \mathbf{Z}$.

We note that condition (ii) is a technical assumption easily fulfilled in the case of a Markov map (by refining the Markov partition if necessary).

Remark 2. Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a Markov family such that $T^n = \tilde{T}^n$ for every $n \in \mathbf{Z}$. If $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ satisfies property (ii) in the case where $c = 1$ then it also satisfies property (ii) for every $c > 1$.

The proof of this remark is in section 4 together with the proof of Remark 1.

2.4. (*Uaa*) *conjugacies from a point to the train track*

The Markov families $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are 'topologically conjugate' if there exists a 'conjugacy family' $(h_n)_{n\in\mathbf{Z}}$ of homeomorphisms $h_n : T^n \to P^n$ such that $h_{n+1} \circ M_n = N_n \circ h_n$ for all $n \in \mathbf{Z}$. The conjugacy family $(h_n)_{n\in\mathbf{Z}}$ is (uaa) if for every n, the homeomorphisms h_n and h^{-1}_n are (uaa) and the moduli of continuity χ_c of h_n and h^{-1}_n do not depend upon n.

Definition 4. Two (uaa) Markov families $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are (uaa) conjugate if there exists a (uaa) conjugacy family between $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$.

An 'orbit' $(w_n)_{n\in\mathbf{Z}}$ of the Markov family $(M_n, T^n_\Delta)_{n\in\mathbf{Z}}$ is a sequence of points $w_n \in T^n$ such that $M_n(w_n) = w_{n+1}$ for every $n \in \mathbf{Z}$. A 'sub-orbit' $(w_{n_i})_{i\in\mathbf{Z}}$ is a sub-sequence of $(w_n)_{n\in\mathbf{Z}}$ (where $(n_i)_{i\in\mathbf{Z}}$ is an increasing sequence of integers).

Theorem 2. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ be (uaa) Markov families, and let $(h_n)_{n\in\mathbf{Z}}$ be a topological conjugacy family between $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$. If, for every point w_{n_i} of a sub-orbit $(w_{n_i})_{i\in\mathbf{Z}}$, h_{n_i} is (aa) at w_{n_i} and the modulus of continuity does not depend upon i, then $(h_n)_{n\in\mathbf{Z}}$ is a (uaa) conjugacy.*

Definition 5. A 'generating set' $\mathcal{G}$ of $(T^n)_{n\in\mathbf{Z}}$ is a set of points $a \in T^{l(a)}$ with $l(a) \in \mathbf{Z}$, and with the property that, for every $n \in \mathbf{Z}$, we have

$$T^n = \mathrm{cl}(\{w = M_{n-1} \circ \cdots \circ M_{l(a)}(a) : a \in \mathcal{G} \quad \text{and} \quad l(a) \le n\}).$$

Theorem 3. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ be (uaa) Markov families, and let $(h_n)_{n\in\mathbf{Z}}$ be a topological conjugacy family between $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$. If, for every point a of a generating set $\mathcal{G}$, $h_{l(a)}$ is (aa) at a and the modulus of continuity does not depend upon a, then $(h_n)_{n\in\mathbf{Z}}$ is a (uaa) conjugacy.*

A 'sub-sequence' $(w_{n_i})_{i\in\mathbf{Z}}$ is any sequence of points $w_{n_i} \in T^{n_i}$ (where $(n_i)_{i\in\mathbf{Z}}$ is an increasing sequence of integers).

Theorem 4. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ be (uaa) Markov families, and let $(h_n)_{n\in\mathbf{Z}}$ be a topological conjugacy family between $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$. If, for every point w_{n_i} of a sub-sequence $(w_{n_i})_{i\in\mathbf{Z}}$, h_{n_i} is (uaa) at w_{n_i} and the modulus of continuity does not depend upon i, then $(h_n)_{n\in\mathbf{Z}}$ is a (uaa) conjugacy.*

We should like to point out that the previous conditions used in the previous theorems correspond to very natural and simple dynamical objects. For instance, take two Markov maps $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$. An example of an orbit $(w_{n_i})_{i\in\mathbf{Z}}$ is a fixed point; an example of a generating set $\mathcal{G}$ is a point with dense orbit; and an example of a sub-sequence $(w_{n_i})_{i\in\mathbf{Z}}$ is a sub-sequence where all points w_{n_i} are the same point.

2.5. *C^r Markov families and C^r conjugacies*

For $r = k + \alpha$, where $k \in \mathbf{N}$ and $0 < \alpha \le 1$, a function $h : I \to \mathbf{R}$ defined on an interval I is C^r if the kth derivative of h is α-Hölder continuous. We say that a function $h : J \to \mathbf{R}$ defined on a set $J \subset \mathbf{R}$ is C^r if h has a C^r extension to an interval $I \supset J$ of $\mathbf{R}$.

An atlas A on a train track T_Δ is C^r if the overlap map between any two charts in A is C^r and its C^r norm is bounded away from infinity independently of the charts considered.

A C^r 'Markov family' $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is a Markov family with the following properties.

(i) The atlases A_n are C^r, and locally the maps M_n with respect to any pair of charts are C^r diffeomorphisms with C^r norm bounded away from infinity independently of the charts considered and of $n \in \mathbf{Z}$.

(ii) There exist constants $c > 0$ and $\lambda > 1$ such that, for all $x \in T^n$ and $p \geq 0$, we have

$$|(j \circ M_{n+p} \circ \cdots \circ M_n \circ i^{-1})'(i(x))| > c\lambda^p$$

where $(i, I) \in A_n$, $(j, J) \in A_{n+p+1}$ and there is an open segment $I' \subset I$ such that $x \in I'$ and $M_{n+p} \circ \cdots \circ M_n(I') \subset J$.

(iii) The property (i) of the definition of the (uaa) Markov family is also satisfied.

A 'Markov map (M, T_Δ, A) is C^r' if there is a C^r Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ with $M_n = M$, $T^n_\Delta = T_\Delta$ and $A_n = A$ for all $n \in \mathbf{Z}$.

The C^r Markov families $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are C^r 'conjugate' if there is a family $(h_n)_{n\in\mathbf{Z}}$ of C^r diffeomorphisms $h_n : T^n \to P^n$ such that $h_{n+1} \circ M_n = N_n \circ h_n$, and the C^r norms of the maps h_n and h_n^{-1} are bounded away from infinity independently of $n \in \mathbf{Z}$.

Theorem 5. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ be C^r Markov families and let $(h_n)_{n\in\mathbf{Z}}$ be a topological conjugacy between $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$. If $(h_n)_{n\in\mathbf{Z}}$ is (uaa) then $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are C^r conjugate.*

2.6. *Canonical charts*

To prove Theorem 5 we shall use the notion of canonical charts and Proposition 1 below. Given a (uaa) Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$, we define a 'canonical chart' (c_0, J_0) with $J_0 \subset T^0$ containing a 1-cylinder as follows (see also Pinto and Rand (1995), Pinto and Sullivan (2001)). Let $I_0, I_{-1}, \ldots$ be segments such that $I_0 = J_0$, $I_m \subset T^m$, $M_m|I_m$ is a homeomorphism on to its image and $M_m(I_m) = I_{m+1}$. Let $K_m : I_m \to J_0$ be the homeomorphism given by $K_m = M_{-1} \circ \cdots \circ M_m|I_m$ for every $m < 0$. Let us denote by j_l and j_r the endpoints of J_0. Take a chart $(i_m, I'_m) \in A_m$ such that $I_m \subset I'_m$. Let $L_m : i_m(I_m) \to (0, 1)$ be the map determined uniquely by $L_m(K_m^{-1}(j_l)) = 0$, $L_m(K_m^{-1}(j_r)) = 1$, and L_m has an affine extension to $\mathbf{R}$. Let $d_m : J_0 \to (0, 1)$ be the chart defined by $d_m = L_m \circ i_m \circ K_m^{-1}$ (see figure 2). By Lemma 4 in section 3.4, the sequence $(d_m)_{m\in\mathbf{Z}}$ converges when m tends to minus infinity. We define the canonical chart $c_0 : J_0 \to \mathbf{R}$ as being this limit $c_0 = \lim_{m\to-\infty} d_m$. The canonical charts (c_0, J_0) with $J_0 \subset T^0$ form the canonical

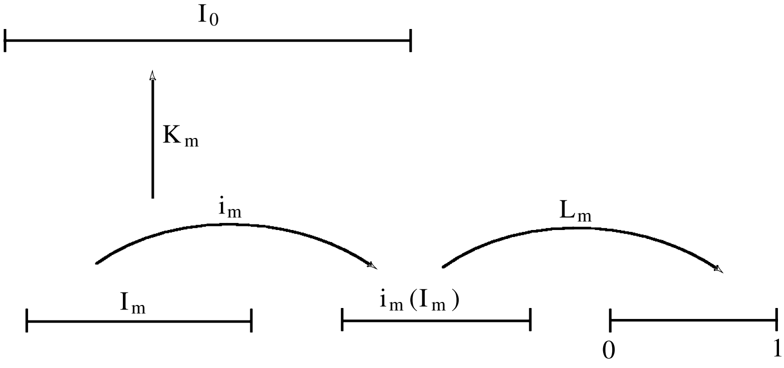

Figure 2. The chart $d_m = L_m \circ i_m \circ K_m^{-1}$.

atlas $C_{A,0}$ on T^0. Similarly, for every $n \in \mathbf{Z}$, we define the canonical charts (c_n, J_n) with $J_n \subset T^n$ containing a 1-cylinder which form the canonical atlas $C_{A,n}$ on T^n.

Proposition 1. *The Markov family $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ attains the maximum possible smoothness in the (uaa) conjugacy class of the Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$. Moreover, the family $(C_{A,n})_{n\in\mathbf{Z}}$ is canonical in the following sense: if $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are (uaa) conjugate by a conjugacy family $(h_n)_{n\in\mathbf{Z}}$ then, for every chart $(c_{A,n}, J_{A,n}) \in C_{A,n}$, there is a chart $(c_{B,n}, J_{B,n}) \in C_{B,n}$ with $J_{B,n} = h_n(J_{A,n})$ such that*

$$r_{c_{A,n}}(x, y, z) = r_{c_{B,n}}(h_n(x), h_n(y), h_n(z))$$

for all distinct points $x, y, z \in J_{A,n}$, or equivalently $c_{B,n} \circ h_n \circ c^{-1}_{A,n}$ has an affine extension to the reals.

Similar results to Proposition 1 are presented in Pinto and Rand (1995), Bedford and Fisher (1997) and Pinto and Sullivan (2001) for $C^{1+\alpha}$ conjugacy classes with $\alpha > 0$ instead of (uaa) conjugacy classes. The proof of this proposition is given in section 3.4.

3. Proof of Theorem 1

Theorem 1 follows from applying Theorems 2–5 to Markov maps. Before proceeding to the proofs of Theorems 2–5, we are going to prove three lemmas that we shall use later in the proofs of these theorems. We note that without loss of generality, we can take the modulus of continuity $\chi_c : \mathbf{R}^+_0 \to \mathbf{R}^+_0$ as being an increasing continuous function. Hence, for simplicity of the arguments in this section we always consider that this is the case.

Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a Markov family. Let $C, D \subset T^n$ be m-cylinders or m-gaps. We say that the sets C and D are 'adjacent' if they have a common endpoint.

Lemma 1. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a (uaa) Markov family. There exists a constant $d > 1$ such that, for all m-cylinders or m-gaps $C, D \subset T^n$ which are adjacent and contained in the domain I of a chart $(i, I) \in A_n$*

$$d^{-1} < \frac{|C|_i}{|D|_i} < d.$$

Proof. Let $C' = M_{n+m-2} \circ \cdots \circ M_n(C)$, $D' = M_{n+m-2} \circ \cdots \circ M_n(D)$, and (j, J) be a chart in the atlas A_{n+m-1} such that $C', D' \subset J$. Let $b > 1$ be as considered in the definition of a (uaa) Markov family. Then

$$b^{-2} < |C'|_j/|D'|_j < b^2. \tag{3}$$

Take $c > b^2$. Using inequaliy (2) and that χ_c is an increasing function, we obtain

$$\left|\log \frac{|C|_i}{|D|_i} \frac{|D'|_j}{|C'|_j}\right| < \chi_c(b). \tag{4}$$

Now, Lemma 1 follows from inequalities (3) and (4). □

Lemma 2. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a (uaa) Markov family. There exist constants $d > 1$ and $0 < \alpha, \beta < 1$ with the property that, for every m-cylinder or m-gap $C \subset T^n$,* *and for all charts* $(i, I) \in A_n$ *such that* $C \cap I \neq \emptyset$, *we have* $|C| < d\beta^m$. *If* $C \subset I$ *then* $|C| > d^{-1}\alpha^m$.

Proof. Since the number of Markov intervals contained in T^n is bounded independently of $n \in \mathbf{Z}$, Lemma 2 follows from Lemma 1. □

Lemma 3. *If* $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ *and* $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ *are two* (*uaa*) *Markov families topologically conjugate by* $(h_n)_{n\in\mathbf{Z}}$ *then they are* C^α *conjugate, for some* $\alpha > 0$, *i.e. there exist constants* $d > 1$ *and* $\alpha > 0$ *with the property that for every chart* $(i, I) \in A_n$, *for all* $x, y \in I$, *and for every chart* $(j, J) \in B_n$ *with* $h_n(x), h_n(y) \in J$, *we have*

$$|h_n(y) - h_n(x)|_j < d|y - x|_i^\alpha \quad \text{and} \quad |y - x|_i < d|h_n(y) - h_n(x)|_j^\alpha. \tag{5}$$

Proof. Let (i, I) be a chart in the atlas A_n, and for all $x, y \in I$ let (j, J) be a chart in B_n such that $h_n(x), h_n(y) \in J$. Then choose the smallest m with the property that there are adjacent m-cylinders or m-gaps C and D, and an $(m+1)$-cylinder or $(m+1)$-gap E such that (i) $x, y \in C \cup D$, and (ii) the interval $K \subset I$ with endpoints x and y contains E. By Lemma 2, there exist constants $d_1 > 1$ and $0 < \alpha_1, \beta_1 < 1$ such that

$$2d_1^{-1}\alpha_1^{m+1} < |E|_i \le |y - x|_i \le |(C \cup D) \cap I|_i < 2d_1\beta_1^m.$$

Similarly, there exist constants $d_2 > 1$ and $0 < \alpha_2, \beta_2 < 1$ such that

$$2d_2^{-1}\alpha_2^{m+1} < |h_n(E)|_j \le |h_n(y) - h_n(x)|_j \le |h_n((C \cup D) \cap I)|_j < 2d_2\beta_2^m.$$

Therefore, there exist constants $d > 1$ and $\alpha > 0$ such that (5) follows. □

3.1. *Proof of Theorem 2*

We are going to prove that the homeomorphism $h_0 : T^0 \to P^0$ is (uaa). Then it follows, in a similar way, that h_n is (uaa) for all $n \in \mathbf{Z}$. For simplicity of exposition, we are also going to consider the case in which the conjugacy is (aa) in an orbit $(w_m)_{m\in\mathbf{Z}}$. The proof for the case where the conjugacy is (aa) just in a sub-orbit follows similarly to this one.

Let (i, I) be a chart in A_0, and $x, y, z \in I$ any three points such that $c^{-1} \le r(x, y, z) \le c$. Take a sequence of charts $(i_m, I'_m) \in A_m$ such that for some $M < 0$ and all $m < M$ it has the following properties: (i) there are intervals I_m and J_m such that $I_m \subset J_m \subset I'_m$, and the maps

$$M_{I_m I} = M_{-1} \circ \cdots \circ M_m : I_m \to I \quad \text{and} \quad M_{J_m J_M} = M_{M-1} \circ \cdots \circ M_m : J_m \to J_M$$

are homeomorphisms; and (ii) $w_m \in J_m \setminus I_m$ (see figure 3). Let x_M, y_M and z_M be the pre-images by $M_{I_M I}$ of x, y and z, respectively. Take a point $p_M \in J_M$ and a constant $\underline{c} = \underline{c}(x_M, y_M, z_M, w_M, p_M) > 1$ such that

$$\underline{c}^{-1} < r(x_M, w_M, p_M), r(y_M, w_M, p_M), r(z_M, w_M, p_M) < \underline{c}$$

(see figure 3). Let $x_m, y_m, z_m, p_m \in J_m$ be the pre-images by $M_{J_m J_M}$ of x_M, y_M, z_M and p_M, respectively. Since the Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is (uaa)

$$\left|\log \frac{r(x, y, z)}{r(x_m, y_m, z_m)}\right| < \chi_c(|z - x|). \tag{6}$$

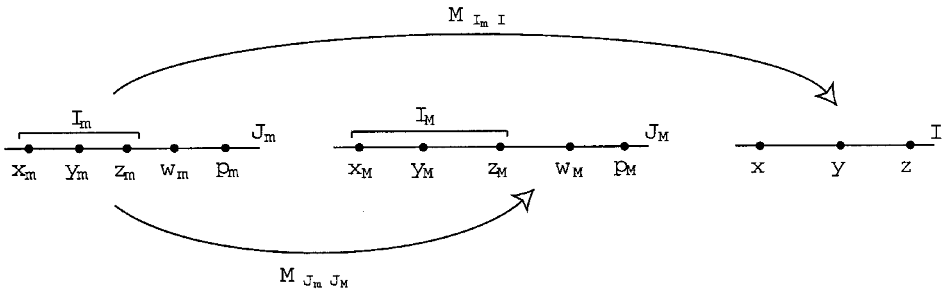

Figure 3. The maps $M_{I_m I}$ and $M_{J_m J_M}$.

Let $(u, U) \in B_0$ and $(u_m, U_m) \in B_m$ be charts such that $h_0(I) \subset U$ and $h_m(J_m) \subset U_m$. Since the Markov family $(N_n, B_n)_{n \in \mathbf{Z}}$ is (uaa) and by Lemma 3, there exist constants $d > 1$ and $0 < \alpha \leq 1$ such that

$$\left| \log \frac{r(h_0(x), h_0(y), h_0(z))}{r(h_m(x_m), h_m(y_m), h_m(z_m))} \right| < \chi_c(|h_0(z) - h_0(x)|_u) < \chi_c(d(|z - x|_i)^\alpha). \quad (7)$$

By hypothesis, the conjugacy h_m is (aa) at w_m, which implies that

$$\left| \log \frac{r(h_m(x_m), h_m(w_m), h_m(p_m))}{r(x_m, w_m, p_m)} \right| < \chi_{\underline{c}}(|p_m - x_m|),$$

$$\left| \log \frac{r(h_m(y_m), h_m(w_m), h_m(p_m))}{r(y_m, w_m, p_m)} \right| < \chi_{\underline{c}}(|p_m - y_m|),$$

$$\left| \log \frac{r(h_m(z_m), h_m(w_m), h_m(p_m))}{r(z_m, w_m, p_m)} \right| < \chi_{\underline{c}}(|p_m - z_m|).$$

The last three inequalities imply that

$$\log \frac{r(x_m, y_m, z_m)}{r(h_m(x_m), h_m(y_m), h_m(z_m))} \to 0, \quad \text{when} \quad m \to -\infty. \quad (8)$$

By (6), (7) and (8), there is a continuous function $\chi'_c : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying $\chi'_c(0) = 0$, and such that

$$\left| \log \frac{r(h_0(x), h_0(y), h_0(z))}{r(x, y, z)} \right| < \chi'_c(|z - x|).$$

Therefore, the conjugacy h_0 is (uaa). □

3.2. *Proof of Theorem 3*

We are going to prove that the homeomorphism $h_0 : T^0 \to P^0$ is (uaa). Then it follows, in a similar way, that h_n is (uaa), for all $n \in \mathbf{Z}$.

Let (i, I) be a chart in A_0, and $x, y, z \in I$ any three points such that $c^{-1} \leq r(x, y, z) \leq c$. By construction of the set $\mathcal{G}$ in section 2.4, there is a sequence $(w_k)_{k \in \mathbf{Z}}$ of points $w_k = (M_{-1} \circ \cdots \circ M_{l(a_k)})(a_k) \in I$ such that (i) $a_k \in \mathcal{G}$, (ii) $i(x) < i(w_k) < i(z)$, and (iii) $\lim w_k = y$. Take a sequence of charts (i_k, I'_k) in $A_{l(a_k)}$ such that for some $K > 0$ and all $k > K$ it has the following properties: (i) there are points $x_k, y_k, z_k, a_k \in I'_k$ whose images by $M_{-1} \circ \cdots \circ M_{l(a_k)}$ are the points

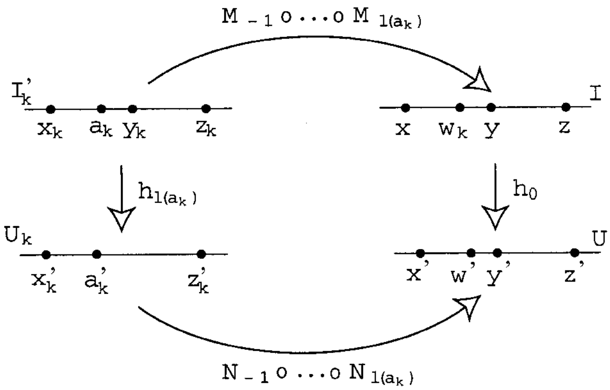

Figure 4. The points in the ratios of the proof of Theorem 3.

$x, y, z, w_k \in I$, respectively; (ii) the interval $I_k \subset I'_k$ with endpoints x_k and z_k contains the points y_k and a_k; and (iii) I_k is sent injectively by $M_{-1} \circ \cdots \circ M_{l(a_k)}$ in the interval with endpoints x and z. Since the Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is (uaa), for k large enough, we get

$$\left|\log \frac{r(x, w_k, z)}{r(x_k, a_k, z_k)}\right| < \chi_c(|z - x|). \tag{9}$$

Set $x' = h_0(x)$, $y' = h_0(y)$, $w'_k = h_0(w_k)$, $z' = h_0(z)$ and $x'_k = h_{l(a_k)}(x_k)$, $a'_k = h_{l(a_k)}(a_k)$, $z'_k = h_{l(a_k)}(z_k)$ (see figure 4). Let $(u, U) \in B_0$ and $(u_k, U_k) \in B_{l(a_k)}$ be charts such that $h_0(I) \subset U$ and $h_{l(a_k)}(I_k) \subset U_k$. Since the Markov family $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ is (uaa) and by Lemma 3, there exist constants $d > 1$ and $0 < \alpha \le 1$ such that

$$\left|\log \frac{r(x', w'_k, z')}{r(x'_k, a'_k, z'_k)}\right| < \chi_c(|z' - x'|_u) < \chi_c(d(|z - x|_i)^\alpha). \tag{10}$$

Since the conjugacy $h_{l(a_k)}$ is (aa) at the point a_k

$$\left|\log \frac{r(x'_k, a'_k, z'_k)}{r(x_k, a_k, z_k)}\right| < \chi_c(|z_k - x_k|). \tag{11}$$

Note that $\chi_c(|z_k - x_k|)$ converges to zero, when k tends to infinity. Therefore, by (9), (10) and (11), there is a continuous function $\chi'_c : \mathbf{R}^+_0 \to \mathbf{R}^+_0$ satisfying $\chi'_c(0) = 0$, and such that

$$\left|\log \frac{r(x', w'_k, z')}{r(x, w_k, z)}\right| < \chi'_c(|z - x|). \tag{12}$$

By continuity of the ratios, we obtain

$$\lim_{k\to\infty} r(x', w'_k, z') = r(x', y', z') \quad \text{and} \quad \lim_{k\to\infty} r(x, w_k, z) = r(x, y, z). \tag{13}$$

Therefore, by (12) and (13), we conclude

$$\left|\log \frac{r(x', y', z')}{r(x, y, z)}\right| \le \chi'_c(|z - x|),$$

and so h_0 is (uaa). □

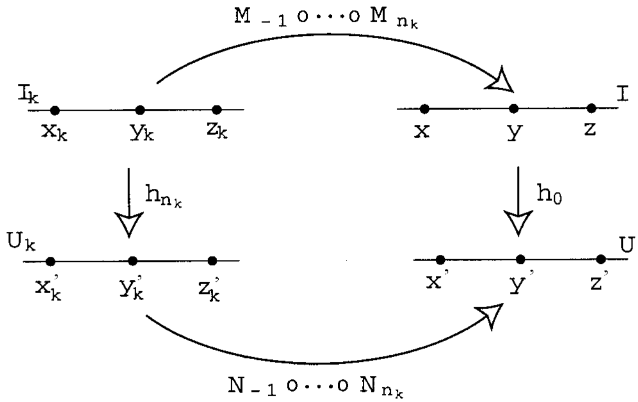

Figure 5. The points in the ratios of the proof of Theorem 4.

3.3. *Proof of Theorem 4*

We are going to prove that the homeomorphism $h_0 : T^0 \to P^0$ is (uaa). Then it follows, in a similar way, that h_n is (uaa) for all $n \in \mathbf{Z}$.

Let (i, I) be a chart in A_0, and $x, y, z \in I$ any three points such that $c^{-1} \leq r(x, y, z) \leq c$. By conditions (v) and (vii) of the definition of Markov partition, there exists $L > 0$ such that for all $n > L$ and all $(n - L)$-cylinders C, we have $(M_{-1} \circ \cdots \circ M_n)(C) = T^0$. Hence, by Lemma 2 there is n_k sufficiently large and there is a chart $(i_k, I_k) \in A_{n_k}$ such that (i) $w_{n_k} \in I_k$; (ii) $|I_k|_{i_k} < |z - x|_i$; (iii) $(M_{-1} \circ \cdots \circ M_{n_k})(I_k) = I$; and (iv) $M_{I_k I} = M_{-1} \circ \cdots \circ M_{n_k} : I_k \to I$ is a homeomorphism. Set $x_k = M_{I_k I}^{-1}(x)$, $y_k = M_{I_k I}^{-1}(y)$ and $z_k = M_{I_k I}^{-1}(z)$ (see figure 5). Since the Markov family $(M_n, T_\Delta^n, A_n)_{n \in \mathbf{Z}}$ is (uaa)

$$\left| \log \frac{r(x, y, z)}{r(x_k, y_k, z_k)} \right| < \chi_c(|z - x|). \tag{14}$$

Set $x' = h_0(x)$, $y' = h_0(y)$ and $z' = h_0(z)$ and $x'_k = h_{n_k}(x_k)$, $y'_k = h_{n_k}(y_k)$ and $z'_k = h_{n_k}(z_k)$. Let $(u, U) \in B_0$ and $(u_k, U_k) \in B_k$ be charts such that $h_0(I) \subset U$ and $h_{n_k}(I_k) \subset U_k$. Since the Markov family $(N_n, P_\Gamma^n, B_n)_{n \in \mathbf{Z}}$ is (uaa) and by Lemma 3, there exist constants $d > 1$ and $0 < \alpha \leq 1$ such that

$$\left| \log \frac{r(x', y', z')}{r(x'_k, y'_k, z'_k)} \right| < \chi_c(|z' - x'|_u) < \chi_c(d(|z - x|_i)^\alpha). \tag{15}$$

Since the conjugacy h_{n_k} is (uaa) at the point w_{n_k} and $|z_k - x_k|_{j_k} < |z - x|_i$, we have

$$\left| \log \frac{r(x'_k, y'_k, z'_k)}{r(x_k, y_k, z_k)} \right| < \chi_c(|z - x|). \tag{16}$$

Therefore, by (14), (15) and (16), there is a continuous function $\chi'_c : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying $\chi'_c(0) = 0$, and such that

$$\left| \log \frac{r(x', y', z')}{r(x, y, z)} \right| < \chi'_c(|z - x|). \tag{17}$$

Therefore, h_0 is (uaa). $\square$

3.4. *Proof of Theorem 5*

Lemma 4. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a (uaa) Markov family, and $(C_{A,n})_{n\in\mathbf{Z}}$ be the family of canonical atlas. The canonical charts $(c_{A,n}, J_n)$ with $J_n \subset T^n$ are well defined and topologically compatible with the charts in A_n for all $n \in \mathbf{Z}$. Furthermore, $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is a (uaa) Markov family, and it is (uaa) conjugate to $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$.*

Proof. Let us begin proving that the canonical chart $(c_{A,0}, J_0)$ with $J_0 \subset T^0$ is well defined by $c_{A,0} = \lim_{m\to-\infty} d_{A,m}$, where the charts $(d_{A,m}, J_0)$ are as introduced in section 2.6. Let x, y, z be any three points in J_0 such that $c^{-1} < r_{d_{A,0}}(x, y, z) < c$. Since the Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is (uaa), the ratios $r_{d_{A,m}}(x, y, z)$ converge to a unique limit $r(x, y, z)$ when m tends to minus infinity. Furthermore

$$\left|\log \frac{r_{d_{A,0}}(x, y, z)}{r(x, y, z)}\right| < \chi_c(|z - x|_{d_{A,0}}). \tag{18}$$

Thus, the ratio $r(x, y, z)$ varies continuously with x, y and z, and there exists a constant $c_1 > 1$ such that $c_1^{-1} < r(x, y, z) < c_1$. Let j_l and j_r be the endpoints of the interval J_0. For every point $y \in J_0$, there is a sequence of pairwise distinct points $x_0, \ldots, x_p, \ldots, x_q \in J_0$ such that $x_0 = j_l$, $x_p = y$, $x_q = j_r$ and $c^{-1} < r_{d_{A,0}}(x_i, x_{i+1}, x_{i+2}) < c$. Hence, writing $r(j_l, y, j_r)$ in terms of the ratios $r(x_i, x_{i+1}, x_{i+2})$ we get that the ratio $r(j_l, y, j_r)$ varies monotonically and continuously with $y \in J_0$. Thus

$$c_{A,0}(y) = \lim_{m\to-\infty} d_{A,m}(y) = \lim_{m\to-\infty} \frac{1}{1 + r_{d_{A,m}}(j_l, y, j_r)} = \frac{1}{1 + r(j_l, y, j_r)},$$

which implies that $c_{A,0}$ is a bijection and topologically compatible with $d_{A,0}$. Hence, the canonical chart $(c_{A,0}, J_0)$ is well defined. Therefore, the set of canonical charts $(c_{A,0}, J_0)$ with $J_0 \subset T^0$ form a topological atlas $C_{A,0}$. Moreover, by inequality (18), the canonical charts $(c_{A,0}, J_0)$ are (uaa) compatible with the charts in A_0.

By a similar construction, for every $n \in \mathbf{Z}$, we obtain that the canonical charts in $C_{A,n}$ are (uaa) compatible with the charts in A_n, and the modulus of continuity does not depend upon the charts considered and upon $n \in \mathbf{Z}$. Therefore, using that the Markov family $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is (uaa), we get that the Markov family $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is also (uaa). □

Lemma 5. *Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ be a $C^{k+\delta}$ Markov family, where $k \in \mathbf{N}$ and $\delta > 0$. Let $(C_{A,n})_{n\in\mathbf{Z}}$ be the family of canonical atlas determined by the family $(A_n)_{n\in\mathbf{Z}}$. Then $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is a $C^{k+\delta}$ Markov family, and $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is $C^{k+\delta}$ conjugate to $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$.*

Proof. We are going to prove that the canonical charts (c_n, J_n) with $J_n \subset T^n$ are $C^{k+\delta}$ compatible with the charts contained in A_n. Furthermore, the overlap maps have $C^{k+\delta}$ norm bounded away from infinity, independently of the charts considered, and of $n \in \mathbf{Z}$.

Let (c_0, J_0) be the canonical chart in $C_{A,0}$ defined by $c_0 = \lim_{m\to-\infty} d_m$, where the charts (d_m, J_0) are given by $d_m = L_m \circ i_m \circ K_m^{-1}$, and the maps L_m, i_m and K_m are as introduced in section 2.6. The map $d_m \circ i_0^{-1}$ is $C^{k+\delta}$ and it is the composition of a contraction $i_m \circ K_m^{-1} \circ i_0^{-1}$ followed by an expansion L_m. Therefore, the $C^{k+\delta}$ norm of the maps $d_m \circ i_0^{-1}$ is uniformly bounded. Hence, using an Arzela–Ascoli argument type there is a sub-sequence of maps $d_{m_l} \circ i_0^{-1}$ that converges in the $C^{k+\delta-\varepsilon}$ norm to a $C^{k+\delta}$ map ψ. Moreover, the $C^{k+\delta}$ norm of ψ is bounded away from infinity independently of the charts (c_0, J_0) and (d_m, J_0) considered. By Lemma 4, the map ψ is equal to $c_0 \circ i_0^{-1}$, where c_0 is the canonical chart. By the same argument, the map $(d_m \circ i_0^{-1})^{-1}$ has a sub-sequence converging in the $C^{k+\delta-\varepsilon}$ norm to a $C^{k+\delta}$ map ϕ, and the $C^{k+\delta}$ norm of ϕ is bounded away from infinity, independently of the charts (c_0, J_0) and (d_m, J_0) considered. By Lemma 4, the map ϕ is equal to $\psi^{-1} = (c_0 \circ i_0^{-1})^{-1}$. Thus, the chart c_0 is $C^{k+\delta}$ compatible with i_0, and the norm of the overlap map ϕ is bounded away from infinity, independently of the charts c_0 and i_0 considered. Similarly, we obtain that the charts (c_n, J_n) with $J_n \subset T^n$ are $C^{k+\delta}$ compatible with the charts contained in A_n and the norm of the overlap maps is bounded away from infinity, independently of the charts considered and of $n \in \mathbf{Z}$.

Therefore, using that $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ is a $C^{k+\delta}$ Markov family, we obtain that $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is also a $C^{k+\delta}$ Markov family, and that $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is $C^{k+\delta}$ conjugate to $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$. □

Proof of Proposition 1. Let $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ be (uaa) Markov families which are (uaa) conjugated by $(h_n)_{n\in\mathbf{Z}}$. Let $(c_{A,0}, J_{A,0})$ with $J_{A,0} \subset T^0$ be a canonical chart contained in $C_{A,0}$ defined by

$$c_{A,0} = \lim_{m\to-\infty} L_{A,m} \circ i_{A,m} \circ K^{-1}_{A,m}$$

where $L_{A,m}$ is an affine map, $(i_{A,m}, I'_{A,m})$ is a chart contained in A_m, and $K_{A,m} = M_{-1} \circ \cdots \circ M_m$ is as defined in section 2.6. Similarly, let $(c_{B,0}, J_{B,0})$ with $J_{B,0} = h_0(J_{A,0})$ be a canonical chart contained in $C_{B,0}$ defined by

$$c_{B,0} = \lim_{m\to-\infty} L_{B,m} \circ i_{B,m} \circ K^{-1}_{B,m}$$

where $L_{B,m}$ is an affine map, $(i_{B,m}, I'_{B,m})$ is a chart contained in B_m, and $K_{B,m} = N_{-1} \circ \cdots \circ N_m$ is as defined in section 2.6. For all distinct points $x, y, z \in J_{A,0}$, let us denote $K^{-1}_{A,m}(x)$, $K^{-1}_{A,m}(y)$ and $K^{-1}_{A,m}(z)$ by x_m, y_m and z_m, respectively. By construction of the charts $c_{A,0}$ and $c_{B,0}$, we have that

$$r_{c_{A,0}}(x, y, z) = \lim_{m\to-\infty} r_{i_{A,m}}(x_m, y_m, z_m) \tag{19}$$

and

$$r_{c_{B,0}}(h_0(x), h_0(y), h_0(z)) = \lim_{m\to-\infty} r_{i_{B,m}}(h_m(x_m), h_m(y_m), h_m(z_m)). \tag{20}$$

Since the family $(h_n)_{n\in\mathbf{Z}}$ is (uaa), there is $\chi_c : \mathbf{R}^+_0 \to \mathbf{R}^+_0$ satisfying $\chi_c(0) = 0$, and such that

$$\left| \log \frac{r_{i_{B,m}}(h_m(x_m), h_m(y_m), h_m(z_m))}{r_{i_{A,m}}(x_m, y_m, z_m)} \right| < \chi_c(|z_m - x_m|_{i_{A,m}}). \tag{21}$$

Putting together (19), (20) and (21), we get

$$r_{c_{A,0}}(x, y, z) = r_{c_{B,0}}(h_0(x), h_0(y), h_0(z)),$$

and so $c_{B,0} \circ h_0 \circ c^{-1}_{A,0}$ has an affine extension to the reals.

Similarly, for every $n \in \mathbf{Z}$ and for all canonical charts $(c_{A,n}, J_{A,n})$ with $J_{A,n} \subset T^n$ and $(c_{B,n}, J_{B,n})$ with $J_{B,n} = h_n(J_{A,n})$, we obtain that $c_{B,n} \circ h_n \circ c^{-1}_{A,n}$ has an affine extension to the reals. Hence, for all distinct points $x, y, z \in J_{A,n}$, $r_{c_{A,n}}(x, y, z) = r_{c_{B,n}}(h_n(x), h_n(y), h_n(z))$.

Let us suppose that the Markov family $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ is C^r, for $r > 1$. By Lemma 5, the Markov family $(N_n, P^n_\Gamma, C_{B,n})_{n\in\mathbf{Z}}$ is C^s, for $s \geq r$. Since the maps $c_{B,n} \circ h_n \circ c_{A,n}^{-1}$ are affine, we obtain that the Markov family $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ is also C^s. Therefore, $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ attains the maximum possible smoothness in the (uaa) conjugacy class of $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$. □

Proof of Theorem 5. By Proposition 1, the Markov families $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, C_{B,n})_{n\in\mathbf{Z}}$ are at least C^r and the conjugacy family between them is as smooth as the Markov families. By Lemma 5, the Markov families $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(M_n, T^n_\Delta, C_{A,n})_{n\in\mathbf{Z}}$ are C^r conjugate, and the Markov families $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, C_{B,n})_{n\in\mathbf{Z}}$ are C^r conjugate. Therefore, $(M_n, T^n_\Delta, A_n)_{n\in\mathbf{Z}}$ and $(N_n, P^n_\Gamma, B_n)_{n\in\mathbf{Z}}$ are C^r conjugate. □

3.5. *Proof of Corollary 1*

It is enough to prove that if a homeomorphism $h : I \subset \mathbf{R} \to J \subset \mathbf{R}$ is differentiable at a point $x \in I$ then h is (uaa) at x. To prove that h is (uaa) at x is equivalent to proving that for every $c \geq 1$ and for every $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all points $y_1, y_2, y_3 \in I$ so that $c^{-1} \leq r(y_1, y_2, y_3) \leq c$ and $0 < |y_1 - x|$, $|y_2 - x|$, $|y_3 - x| < \delta_0$, we have that

$$\left|\log \frac{h(y_2) - h(y_1)}{h(y_3) - h(y_2)} \frac{y_3 - y_2}{y_2 - y_1}\right| < \varepsilon. \tag{22}$$

If h is differentiable at a point x then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in [x - \delta, x + \delta]$ we get

$$|\log(h(z) - h(x))(z - x)^{-1}(h'(x))^{-1}| < \varepsilon/4.$$

By continuity of h, for all $z \in [x - \delta, x + \delta]$ there exist $\delta_z, \delta'_z > 0$ such that for all $w \in [z - \delta_z, z + \delta_z]$ and $y \in [x - \delta'_z, x + \delta'_z]$ we obtain that

$$|\log(h(w) - h(y))(w - y)^{-1}(h'(x))^{-1}| < \varepsilon/2.$$

By compactness of $[x - \delta, x + \delta]$ there is a finite set $\{z_1, \ldots, z_n\}$ of points in $[x - \delta, x + \delta]$ such that

$$[x - \delta, x + \delta] \subset \bigcup_{i=1}^{n} [z_i - \delta_{z_i}, z_i + \delta_{z_i}].$$

Hence, for all $\varepsilon > 0$ there exists $\delta' = \min\{\delta, \delta'_{z_1}, \ldots, \delta'_{z_n}\}$ such that for all $w \in [x - \delta', x + \delta']$ and $y \in [x - \delta', x + \delta']$ we get

$$|\log(h(w) - h(y))(w - y)^{-1}(h'(x))^{-1}| < \varepsilon/2.$$

Thus, for all points $y_1, y_2, y_3 \in I$ so that $0 < |y_1 - x|, |y_2 - x|, |y_3 - x| < \delta'$, we get

$$\left|\log\frac{h(y_2)-h(y_1)}{h(y_3)-h(y_2)}\frac{y_3-y_2}{y_2-y_1}\right| \le |\log(h(y_2)-h(y_1))(y_2-y_1)^{-1}(h'(x))^{-1}|$$
$$+ |(h(y_2)-h(y_1))^{-1}(y_2-y_1)h'(x)|$$
$$< \varepsilon,$$

which implies (22) and so ends the proof (in particular, the modulus of continuity χ_c does not depend upon $c \ge 1$). □

4. Proof of Remarks 1 and 2

Proof of Remark 1. Follows similarly to the proof of Remark 2. □

Proof of Remark 2. Let us prove that, for every $c \ge 1$ and for all small $\varepsilon > 0$, there exists $\delta = \delta(c, \varepsilon)$ such that, for all maps $M_{IJ} : I \to J$, for all charts $(i, I') \in A_m$ and $(j, J') \in A_n$ with $I \subset I'$ and $J \subset J'$, there exists $\delta_0 = \delta_0(c, \varepsilon)$ such that, for all $\delta < \delta_0$ and for all points $x, y, z \in J$ with $c^{-1} \le r(x, y, z) \le c$ and $0 < j(y) - j(x)$, $j(z) - j(y) < \delta$, we have

$$\left|\log\frac{r(M_{IJ}^{-1}(x), M_{IJ}^{-1}(y), M_{IJ}^{-1}(z))}{r(x, y, z)}\right| < \varepsilon, \tag{23}$$

and so M is (uaa).

Let us denote by $[t]$ the integer part of $t \ge 0$. There exists $k = k(c, \varepsilon)$ such that, for every pair of adjacent intervals $L, R \subset \mathbf{R}$ with $c^{-1} < |L|/|R| < c$, there are adjacent intervals $P_1, \ldots, P_k$ and a constant $l = l(L, R)$ with the following properties (see figure 6):

(i) $\bigcup_{i=1}^{l-1} P_i \subset L$, $\bigcup_{i=l+1}^{k} P_i \subset R$ and $\bigcup_{i=1}^{k} P_i = L \cup R$;
(ii) $|\log|L|/|\bigcup_{i=1}^{l} P_i|| < \frac{\varepsilon}{3}$ and $|\log|R|/|\bigcup_{i=l+1}^{k} P_i|| < \frac{\varepsilon}{3}$.

Thus, there exist constants $k = k(c, \varepsilon)$ and $l = l(j(x), j(y), j(z))$ and points $x_1, \ldots, x_{k+1} \in J$ with the following properties:

(i) $x_1 = x$ and $x_{k+1} = z$;
(ii) the intervals $[j(x_1), j(x_2)], \ldots, [j(x_k), j(x_{k+1})]$ have the same length and pairwise disjoint interiors;

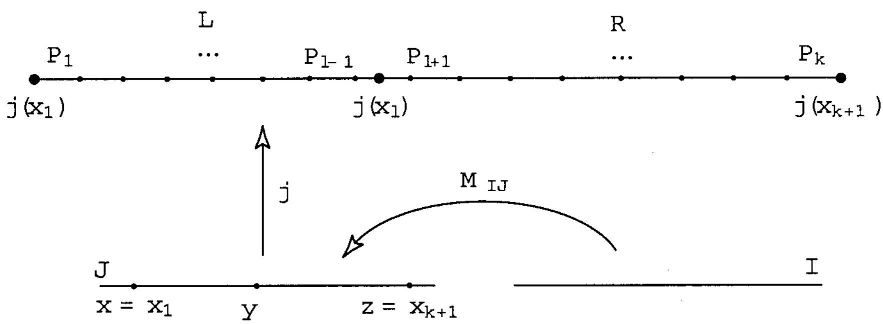

Figure 6. The intervals P_i.

$$\text{(iii)} \qquad \left|\log \frac{j(x_l)-j(x_1)}{j(y)-j(x)}\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\log \frac{j(x_{k+1})-j(x_{l+1})}{j(z)-j(y)}\right| < \frac{\varepsilon}{3}. \tag{24}$$

For simplicity of notation, let us denote the map $i \circ M_{IJ}^{-1}$ by f. Since, by hypotheses $(M_n, T_\Delta^n, A_n)_{n \in \mathbf{Z}}$ satisfies property 1 with $c = 1$ in the definition of a (uaa) Markov family, there is a continuous function $\chi_1 : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying $\chi_1(0) = 0$ such that, for all $1 < p < k+1$

$$\left|\log \frac{f(x_p)-f(x_{p-1})}{f(x_{p+1})-f(x_p)}\right| < \chi_1(\delta).$$

Thus, for all $1 \leq n < k+1$ and $1 < m \leq k+1$

$$\left|\log \frac{f(x_m)-f(x_{m-1})}{f(x_{n+1})-f(x_n)}\right| < |m-n|\chi_1(\delta) < k(c,\varepsilon)\chi_1(\delta),$$

and so there exists a constant $c_1 > 0$ such that

$$\begin{aligned}(1 - c_1 k(c,\varepsilon)\chi_1(\delta))|f(x_n)-f(x_{n-1})| &< |f(x_m)-f(x_{m-1})| \\ &< (1 + c_1 k(c,\varepsilon)\chi_1(\delta))|f(x_n)-f(x_{n-1})|.\end{aligned}$$

Therefore, there exists a constant $c_2 > 0$ such that

$$\left|\log \frac{\sum_{p=1}^{l}(f(x_{p+1})-f(x_p))}{\sum_{p=l+1}^{k}(f(x_{p+1})-f(x_p))}\frac{k-l}{l}\right| \leq c_2 k(c,\varepsilon)\chi_1(\delta). \tag{25}$$

Let us choose $\delta_0 > 0$ such that for all $\delta < \delta_0$ we get $c_2 k(c,\varepsilon)\chi_1(\delta) < \varepsilon/3$. By inequalities (24) and (25), we obtain that

$$\left|\log \frac{r(M_{IJ}^{-1}(x), M_{IJ}^{-1}(y), M_{IJ}^{-1}(z))}{r(x,y,z)}\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon,$$

which ends the proof. □

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