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# Referenda outcomes and the influence of polls: a social network feedback process

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This paper proposes a model to explain the differences between outcomes of referenda and the voting trends suggested by polls. Two main effects are at stake. First, the evolution of the voters' attitudes is conditional on the public information made available to them. Second, the predisposition toward abstention among individuals within each voting group may be different. Our model describes how these two aspects of decision making may interact, showing how publicly available information may amplify the distinct tendency toward abstention between both groups and thus affect the outcome of the referendum.

# 1 Introduction

Voting is an intrinsic part of the modern concept of democracy and the desire to anticipate the polling results is every bit as ancient as the very notion of ballot casting. Polls sample information from the voting population to infer the dynamic evolution of the voters' intention until the date of the elections. On the one hand polls are important ingredients for political strategists to center their efforts and resources in order to increase their share of votes. On the other hand some political analysts fear that polls may influence the outcome of the process. In many countries there are laws in place to prevent polling results from becoming publically available close to the election days, reflecting this concern. Also, several analysts claim that unexpected abstention may result from the perception of a clear winner based on polls, relaxing the motivation of individuals to vote. The level of abstention may clearly impact polls' ability to predict the results of an election accurately.

We argue that although polls may accurately unveil the political preferences of the voters, sometimes they do a poor job of predicting the voters' willingness to vote on election day. And yet, most of the time accuracy seems to be the dominant view; the Gallup polling organization, for example, has often been accurate in predicting the outcome of U.S. Presidential elections and the margin of victory for the election winner. However, it was off the mark in some close elections, such as in 1948 (predicting the victory of Republican Dewey over Democrat Truman), 1976 (a victory for Republican Gerald Ford over Democrat Jimmy Carter), and 2000 (Democrat Al Gore over Republican George W. Bush). In 1982 Los Angeles Mayor Tom Bradley lost the California gubernatorial race despite being ahead in voter polls going into the elections.

Such errors are also observed outside the U.S. In the 1992 U.K. general election the final opinion polls gave the Conservatives between 38% and 39% of the vote, about 1% behind the Labour Party - suggesting that the election would produce a hung parliament or a narrow Labour majority and end 13 years of Tory rule. In the final results, the Conservatives had a lead of 7.6% over Labour and won their fourth successive general election. In Spain's 2004 general elections the conservative PP was expected to maintain its parliamentary majority but the Socialist PSOE secured 164 seats to the PP's 148. In Portugal, a recent analysis of pre-

election accuracy of polls (see Magalhaes, 2005) finds it comparable to that in other developed countries, suggesting that errors are caused by either the closeness of elections or a shared inability of pollsters to deal appropriately with the problems caused by low turnout (turnout in national elections has fallen from 98% in 1975 to less than 50% in recent years).

This phenomenon applies not only to general elections but also to other voting processes, such as referenda. In the recent past there have been a number of referenda in Portugal about several different political and social issues. The first referendum about the legalization of abortion was held in 1998 and a sequence of polls pointed out that the position in favor of legalization would win (see Fig. 1). Among 11 different polls (including some associated with the Catholic Church) during a period of two months, values ranged from 52% to 86% in favor of legalization.

[Insert Figure 1 here]

The surprising outcome of the referendum was a majority of valid votes (50.9%) against the legalization. However, far less than 50% of the population voted - only 31.9% - making the result legally non-binding. The experts argued that the repeated announcements of victory from “yes camp” reinforced the abstention clustering among the part of the population in favor of the legalization. On the contrary, people against legalization of abortion mobilized to vote.

Since the result was non-binding, a new referendum was called in 2007 (see Fig. 2). Although in the first referendum the opinion makers for legalization had focused their efforts on the discussion of the issue per se, in the second referendum their effort was complemented with a strong appeal to participation in the voting process. In the second referendum 59.25% of the valid votes were in favor and 40.75% were against, with a participation of 44% of voters, this time confirming the polls.

[Insert Figure 2 here]

In many of the cases described above, polling agents failed because absenteeism was considerable. But why was absenteeism not an issue in the other instances? To answer this question, we build a model assuming a feedback effect in the way polls influence results, and argue that two main effects are at stake. First, the information made public is possibly insufficient to at-

tain the optimal individual decision. By public information we understand both the perception about the intentions of other voters, and also the information resulting from the public debate about the different issues at stake in the voting process. In a general election, noise is clearly generated by both, since public debate is spread across different topics. In the case of referenda, however, there is only one issue at stake and noise affecting the individual decisions can only arise from the perception of voting intentions. In that sense, by applying our model to the case of referenda, we assume that incomplete information prevents voters from incorporating the expected abstention level in the update of their optimal voting decision, possibly preferring to abstain when they should ideally vote, or preferring to vote when they should ideally abstain. Second, there may be an asymmetric predisposition toward abstention among the individuals willing to vote for each of the different possible outcomes. Given that such a predisposition may be amplified by social interactions and the fact that the publicly available information of polls is incomplete, there is an interaction of these two features of decision making, possibly leading to stronger deviations from the forecast. This may lead not only to significant errors in the winning margin forecast but also to a reversed result, as in the Portuguese referenda example discussed above.

Our main research question is thus to understand how polls influence the final voting of individuals and may also ultimately affect the abstention level and their own forecast ability. The model answers to a non-trivial tradeoff faced by voters, namely how to take into account this feedback effect stemming from incomplete information while reducing the risk of having their stand misrepresented in the final voting result. Such strategic behavior leads some voters to incur the cost of voting with an eye to the benefits of the final result. This model has an additional appealing feature, namely the fact that it can be easily calibrated. Thus, it can be used to forecast results and be appropriately tested in a comprehensive way, allowing analysts to check their interpretation of surprising results.

Our paper provides an empirical application by calibrating the model for the above-mentioned case of the abortion referendum in Portugal. We consider an election with an arbitrarily large number of voters and two voting alternatives with a simple majority decision rule. Voting is not

compulsory, and information about the voting topic is widespread as a result of public debates and the agency of news media. Additionally, a sequence of public polls shows how the opinion of the electorate regarding the issue at stake trends over time. The choice of this referendum for our analysis is based on two reasons: first, the data were easily accessible, and second, and more importantly, there have been two referenda on the same topic - which is a rare occurrence. This allowed for two different calibrations of the model, in the first referendum and the second a few years later. By comparing the different parameters obtained from the two calibrations, we may infer how the results affected the predisposition of voters to vote and abstain in the second referendum. We are thus able to explain why the results of the first referendum were so distant from forecasts and why the results of the second referendum confirmed all the forecasts, corroborating much of the political analysts' belief. As stressed above, our model builds on two points: the incomplete flow of information provided by the sequence of polls and the social interactions that may amplify the misreading of these polls, leading to erroneous forecasts. The argument to build such a model must come from the understanding of how individuals, on the basis of polls, update their decision on why and how to vote at all.

The remainder of the paper is organized as follows. We first finish this section with a brief review of the literature. In the next section we lay out the model, followed by a section in which the results are presented and their implications analyzed. We then calibrate a linearized version of the model for the case of both abortion referenda in Portugal. The outcome of the calibration is analyzed in light of the arguments raised by the political analysts, strongly validating them. The last section presents our conclusions.

## 1.1 Literature review

The benchmark for a modern understanding of the issue why people vote is the seminal work of Downs (1957). If there is a cost for voting and a virtually zero probability that one's vote is decisive, no individual would have an incentive to vote – and yet people continue to vote. A more modern perspective suggests that people vote strategically, i.e. individuals decide why and how to vote based on information about how other agents intend to vote, as mentioned in

recent reviews such as Dhillon and Peralta (2002), Feddersen (2004), and Gheys (2006). This is supported by empirical evidence reported in Cox (1997), Aldrich (1993), and Blais (2000, 2003), among others. Also a relatively recent literature (e.g. Klor and Winter, 2008, Grosser and Schram, 2010, Knight and Schiff, 2010) provides empirical and experimental evidence of the impact of polls in turnout and results of elections. By revealing updated information about the winning likelihood of the different platforms, some candidates gain momentum and others lose it, a mechanism that ends up giving much more comparative weight to the earliest polls than to the last ones.

In order to solve the original voting issue, Riker and Ordeshook (1968) evolved from Downs' (1957) and Tullock's (1967) subsequent work to suggest a consumption benefit from voting that, as pointed by Ferejohn and Fiorina (1975), would assume away the voting paradox. In this context, the decision to vote compares benefits and costs, taking into account the probability of being pivotal. Ledyard (1981, 1984) introduced a game-theoretical approach, making this probability endogenous, although ignoring voting costs. Palfrey and Rosenthal (1983, 1985) introduced costs in that game and concluded that when there is uncertainty about the actual number of voters, high turnout equilibria would not exist, since the probability that a vote is pivotal could be shown to be very small as the size of the voting population increases. Within the game-theoretical approach, Feddersen and Pesendorfer (1996, 1999) take an alternative route trying to understand why voters could possibly abstain under costless voting, and find that voters with strict preferences might abstain due to asymmetric information. This result does not explain the non-voting paradox, but brings the critical role of information to the decision-making process.

Another branch of the voting literature has focused on explaining the consumption premium for voting, and converged to two classes of group-based models, the mobilization models and the ethical voter models. The first assumes that voters are coordinated and rewarded by leaders (Uhlaner, 1989; Morton, 1987 and 1991; Shachar and Nalebuff 1999, Herrera and Martinelli, 2006), while the latter assumes that individuals vote because they are motivated by altruistic concerns for the welfare of others in their group of reference (Gooding and Roberts, 1975;

Harsanyi, 1977 and 1992; Kinder and Kiewiet, 1979; Markus, 1988; Coate and Conlin 2004; Feddersen and Sandroni 2006). Such group-based models have found empirical support (e.g. Schram, 1991 and 1992; Schram and Sonnenmans, 1996a,b; Hill and Leighley, 1996; Coate and Conlin, 2004), but cannot ensure (except for some very particular examples) the existence of equilibria or provide an endogenous mechanism that explains why an individual would join or be identified with any specific group, as pointed out by Feddersen (2004).

## 2 Model

Our model combines the strategic view of voting behavior with the feedback impact and is also based on a group-based reward, in the sense that we analyze aggregate behavioral outcomes (election turnouts together with the choice of the winner) when the behavior of voters exhibits social interaction effects. We follow the approach proposed by Brock and Durlauf (2001a, b) for the case of large elections. Individuals' utilities are composed of a private utility associated with a choice and a social utility that reflects the desire to conform to others' behavior in a non-cooperative decision-making setting. What is distinctive in this approach is that an individual's choice is affected by the beliefs that the voter has about others' intentions, as inferred from the polls' results. The social interaction component of this model is thus reflected in the influence of the average of all the other voters' attitudes, as expressed by the polls, on one's own voting attitude. The choice model is closed by imposing self-consistency between the subjective beliefs and the objective probabilities conditional on the information made available by the polls, implying rational expectations. Brock and Durlauf (2001a,b) not only discuss the existence and stability of multiple equilibria in this model, but also show that in the presence of these conformity effects, the decentralized individual decision-making maximizes the aggregate social welfare. All such results are extended to a multinomial setting in Brock and Durlauf (2002, 2006), which is here suitably adapted and extended for the election model.

For a given information set, Brock and Durlauf's approach provides the steady-state equilibrium for the voting decision. However, as polls are disclosed over time, the information is

updated and the voting attitude of each individual is revised taking into account not only his or her private values, but also his or her desire to conform to others' aggregate intended behavior. The voting decision in our model is taken as a result of the tradeoff of these effects. We assume that all voters' attitudes are revised under the same updated information set and can prove that the final result of the process will converge to one of several possible multiple equilibria.

The comparative statics of our model are consistent with well-known empirical facts. Under complete information, the larger the lag between the proportion of favorable and unfavorable voters, the more comfortable the voters for the winning position will be, and the less important each of them will feel as a voter. The larger the announced gap, the less pivotal each voter feels, and the more willing he or she is to change his or her voting attitudes to abstention. On the contrary, the closer the gap in the polls, the less willing the voters are to abstain, since each voter becomes more pivotal. We show that under incomplete information this discrepancy between the forecast of the polls and the final results is aggravated, possibly inducing the type of reversed observed outcomes.

## 2.1 Modeling individual choice with private social utility

The electorate is modeled as a system with a large number  $I$  of social agents, where each agent  $i$  has to decide on a voting attitude to adopt  $\omega_i$  (e.g.  $\omega_i = 1$  in favor,  $\omega_i = 2$  to abstain, and  $\omega_i = 3$  against) by solving an individual utility maximization problem at some common time. The space of all possible set of actions or decision outcomes taken by the population is the  $N$ -tuple  $\omega = (\omega_1, \dots, \omega_N)$ . Also,  $\omega_{-i} = (\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_N)$  denotes the set of the choices of all other agents except for the agent  $i$  itself. The individual utility,  $V_i(\omega_i)$ , follows the structure proposed by Brock and Durlauf (2001a) and is assumed to consist of three components

$$V_i(\omega_i; \mu_i^e, \epsilon_i) = U(\omega_i) + S(\omega_i, \mu_i^e) + \epsilon_i(\omega_i). \quad (1)$$

The choice variable for agent  $i$  is the voting attitude  $\omega_i$ , which can take any value of the support set  $\{1, 2, 3\}$ , chosen by each individual so as to maximize the expected utility in (1)

under the conditional probability measure  $\mu_i^e$ . The term  $\mu_i^e \equiv \mu_i^e(\omega_{-i})$  denotes the conditional probability measure that agent  $i$  places on the choices of the others at the time of making his or her decision. We shall denote by  $\omega_i^*$  the actual outcome of the decision process of agent  $i$ ,  $\omega_i^* = \operatorname{argmax} V_i(\omega_i)$ . The utility function of each agent takes into account not only individual preferences expressed in the first term  $u_i(\omega_i)$ , but also information about the voting attitudes of the other agents, expressed in the term  $S_i(\omega_i, \mu_i^e)$  and obtained from the publicly available results of polls, which generate  $\mu_i^e$ . The term  $\epsilon_i \equiv \epsilon_i(\omega_i)$  is a random contribution to the utility that is observable to each agent  $i$  but unobservable to the modeler, and accounts for the heterogeneity of the different agents for each possible decision outcome  $\omega_i$ .

The voting attitudes are expressed as an overt behavior in two types of situations: when agents reply to poll queries and when they actually vote. It is assumed that polling takes place at an instant  $t$  prior to the decision process, which occurs at instant  $t + 1$ , and that the public announcement of the results of the polls drives the agents to reevaluate their existing voting attitude. The outcome of the decision process is reflected only in later polls, say at instant  $t + 2$ . A difference in poll outcomes at instants  $t$  and  $t + 2$  results from a reevaluation of individual voting attitudes at  $t + 1$ , and so on. This describes a dynamic process in which the agents change their voting attitude until the referendum day, when their decision becomes definite.

From a collective point of view, the state of the system is described by the fraction of agents adopting the voting attitudes  $\omega_i = 1, 2$ , and  $3$ , respectively  $\Delta_1$ ,  $\Delta_2$ , and  $\Delta_3$  - or in a compact vector form  $\mathbf{\Delta} \equiv (\Delta_1, \Delta_2, \Delta_3)$ . By far most agents have no direct access to information about the state of the system except through the results of the polls. These may introduce some distortion due not only to limitations in the sampling or polling methodologies, but also because the media that publicize the results may not provide a complete analysis of that information. For example, many media tend to focus on the relative percentage of agents in favor of or against, and neglect the level of abstention. The results of the polls are designated by  $m_1$ ,  $m_2$ , and  $m_3$  - or in a compact vector form  $\mathbf{m} \equiv (m_1, m_2, m_3)$  -, corresponding to the frequencies of the statistical classes used in the pools, and the result of which is made public. If the information made available to the public about the results of the pools is incomplete, for example lacking

the fraction associated with abstention, the corresponding value of  $m_2$  is null, implying that this information has no impact on the decision of the agent. Under these considerations, the dependence of the individual utility function  $U_i$  on  $\mu_i^e$  is formally replaced by a dependency on  $\mathbf{m}$ , and it is thus possible to write

$$V_i(\omega_i) = U(\omega_i) + S(\omega_i, \mathbf{m}) + \epsilon_i(\omega_i). \quad (2)$$

The first term denotes a deterministic private utility and is given by

$$U(\omega_i) = \sum_{\alpha=1}^3 u_{\alpha} \delta(\omega_i, \alpha) = \begin{cases} u_1 & \text{if } \omega_i = 1 \\ u_2 & \text{if } \omega_i = 2 \\ u_3 & \text{if } \omega_i = 3 \end{cases},$$

thereby describing the individual preferences of each agent independently of the impact of the results of the polls. The constants  $u_{\alpha}$ , with  $\alpha = 1, 2$  and  $3$ , correspond to the expected individual utility when the agent chooses the voting attitude  $\alpha = \omega_i$ , ignoring the polls. Also, the Kronecker delta is defined according to

$$\delta(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}.$$

The second term of equation (2) denotes a deterministic social utility that models the strategic rationality of the agent when confronted with the results of the polls  $\mathbf{m}$ . This term is given by

$$S(\omega_i, \mathbf{m}) = \sum_{\alpha=1}^3 s_{\alpha}(\mathbf{m}) \delta(\omega_i, \alpha) = \begin{cases} s_1(\mathbf{m}) & \text{if } \omega_i = 1 \\ s_2(\mathbf{m}) & \text{if } \omega_i = 2 \\ s_3(\mathbf{m}) & \text{if } \omega_i = 3 \end{cases},$$

thus describing the social interaction or pressure between agents mediated by the results of

polls. The functions  $s_\alpha(\mathbf{m})$  with  $\alpha = 1, 2, 3$  describes how the expected utility of an agent depends on the results of the polls if the agent adopts the attitude  $\alpha = \omega_i$ . The deterministic part of the utility function is

$$\bar{V}(\omega_i, \mathbf{m}) = U(\omega_i) + S(\omega_i, \mathbf{m}) \quad (3)$$

and can be written in a more compact and useful way as

$$\bar{V}(\omega_i, \mathbf{m}) = \sum_{\alpha=1}^3 f_\alpha(\mathbf{m}) \delta(\omega_i, \alpha) = \begin{cases} s_1(\mathbf{m}) + u_1 & \text{if } \omega_i = 1 \\ s_2(\mathbf{m}) + u_2 & \text{if } \omega_i = 2 \\ s_3(\mathbf{m}) + u_3 & \text{if } \omega_i = 3 \end{cases} \quad (4)$$

where  $f_\alpha(\mathbf{m}) \equiv s_\alpha(\mathbf{m}) + u_\alpha$  is the utility specific for each potential voting decision  $\alpha$ .

The third term in equation (2)  $\epsilon_i(\omega_i)$  denotes a random private utility, which is different for each agent, thus modeling agent heterogeneity, and has the following structure

$$\epsilon_i(\omega_i) = \sum_{\alpha=1}^3 \epsilon_{i,\alpha} \delta[\omega_i, \alpha] = \begin{cases} \epsilon_{i,1} & \text{if } \omega_i = 1 \\ \epsilon_{i,2} & \text{if } \omega_i = 2 \\ \epsilon_{i,3} & \text{if } \omega_i = 3 \end{cases},$$

where  $\epsilon_{i,\alpha}$  (with  $\alpha = 1, 2$ , and  $3$ ) are random independent variables, not necessarily equally distributed, the realizations of which may depend on the potential adopted attitude  $\omega_i$ .

Hence, the complete individual utility function becomes

$$V_i^{t+1}(\omega_i, \mathbf{m}^t) = \sum_{\alpha=1}^3 [f_\alpha(\mathbf{m}^t) + \epsilon_{i,\alpha}] \delta[\omega_i, \alpha] = \begin{cases} f_1(\mathbf{m}) + \epsilon_{i,1} & \text{if } \omega_i = 1 \\ f_2(\mathbf{m}) + \epsilon_{i,2} & \text{if } \omega_i = 2 \\ f_3(\mathbf{m}) + \epsilon_{i,3} & \text{if } \omega_i = 3 \end{cases} \quad (5)$$

where the superscripts  $t$  and  $t + 1$  are introduced to indicate at which instant of the dynamics

each quantity should be evaluated.

Each agent  $i$  chooses the alternative  $\omega_i^*$  that provides his or her maximum utility, say

$$\omega_i^{*t+1} = \underset{\omega_i=1,2,3}{\operatorname{argmax}} V_i^{t+1}(\omega_i, \mathbf{m}^t).$$

Given that the individual utility function has a random component  $\epsilon_i$ , one may only compute the probability of each agent adopting an attitude  $\omega_i^* = \omega_i$ , say

$$P(\omega_i^{*t+1} = \omega_i) = P[V_i^{t+1}(\omega_i^*, \mathbf{m}^t) \geq V_i^{t+1}(\omega_i, \mathbf{m}^t), \forall \omega_i = 1, 2, 3].$$

## 2.2 Pure herding effect

Consider the process of individual decision-making within a group, leading to a possibly different behavior for each member. We characterize a herd when the majority of the resulting behaviors are largely aligned, in spite of possible different personal preferences. In such a case individuals in the group act together without any planned direction, presenting a collective, relatively homogeneous, behavior. In the context of our decision-making model we would say that herd behavior occurs when the peer pressure described by the social utility part of equation (2) is greater than the private preferences as described by the first term. In this section we consider that the social utility component will reflect the gain of utility for an individual by agreeing with the choice of the other elements in the group.

In order to capture the herding effect, an increase of the utility of voting for a given alternative  $\alpha$  must follow as the polls reveal a larger fraction of voters in that same alternative  $\alpha$ . In other words, the function  $f_\alpha$  must be an increasing function of the fraction  $m_\alpha$  of agents assumed to adopt the position  $\alpha$ . For simplicity, the social interaction utility component describing the herding effect  $s_\alpha^h(\mathbf{m}^t)$  is assumed to be linear, say

$$s_\alpha^h(\mathbf{m}) = a_{\alpha\alpha} m_\alpha, \tag{6}$$

where  $a_{\alpha\alpha}$  are some positive constant parameters to guarantee the increasing behavior. Then,

it follows that  $f_\alpha^h(\mathbf{m}) = a_{\alpha\alpha}m_\alpha + u_\alpha$  and that

$$V_i(\omega_i, \mathbf{m}) = \sum_{\alpha=1}^3 [a_{\alpha\alpha}m_\alpha + u_\alpha + \epsilon_{i,\alpha}] \delta(\omega_i, \alpha). \quad (7)$$

## 2.3 Cross-herding effect

We now lead on to a more sophisticated version of the herding effect than the one considered above. Here we will consider that the decision of each agent is not only reinforced by the number of agents that agree with him or her, but also can be affected by the foreseen level of abstention. For example, agents who might prefer to abstain in the absence of the polls can be driven to conform to the winning side once the polls are public, thus reinforcing that position. This can also be captured by adopting a linear social interaction utility component, but now with a more complex structure, say

$$s_\alpha^h(\mathbf{m}) = \sum_{\beta=1}^3 a_{\alpha\beta}m_\beta, \quad (8)$$

where the diagonal terms  $a_{\alpha\alpha}$  (i.e. with  $\alpha = \beta$ ) correspond to the pure herding effect described in the previous section, and the off-diagonal terms  $a_{\alpha\beta}$  (with  $\alpha \neq \beta$ ) describe the social pressure between different voting attitudes, denoted here as a cross-herding effect. In this case the value of  $a_{\alpha\beta}$  can be interpreted as the marginal increase in utility for an agent willing to vote for alternative  $\alpha$  if he or she aligns with voting for alternative  $\beta$ , reflecting a strategic voting behavior. The matrix  $a$  can thus be seen as a payoff matrix, leading each player to adopt a voting strategy that considers the aggregate decision of all other players. In this sense the herding effect with  $a_{\alpha\alpha} > a_{\alpha\beta}$  for  $\alpha \neq \beta$  can be understood as a coordination game.

It now follows that  $f_\alpha^h(\mathbf{m}) = \sum_{\beta=1}^3 a_{\alpha\beta}m_\beta + u_\alpha$  and

$$V_i(\omega_i, \mathbf{m}) = \sum_{\alpha=1}^3 \left[ \sum_{\beta=1}^3 a_{\alpha\beta}m_\beta + u_\alpha + \epsilon_{i,\alpha} \right] \delta(\omega_i, \alpha). \quad (9)$$

The interesting aspects of the cross-herding is that the levels of abstention can impact

differently on the sides of the “yes” and of the “no”, caused by an asymmetry in the parameters  $a_{12}$  and  $a_{23}$  (or  $a_{21}$  and  $a_{32}$ ). As a consequence, one of the voting sides (“yes” or “no”) can drag more undecided voters to their stand than the other, thus affecting the outcome of the polls and of the voting process.

## 2.4 From individual decisions to polls

The public information resulting from the polls at time  $t$  is described by the vector  $\mathbf{m}^t = (m_1^t, m_2^t, m_3^t)$  and is obtained by sampling the voting attitudes  $\Delta^{t-1} = (\Delta_1^{t-1}, \Delta_2^{t-1}, \Delta_3^{t-1})$  from the population at time  $t - 1$ . In a formal way, this is summarized by

$$\mathbf{m}^t = R(\Delta^{t-1}), \quad (10)$$

where  $R$  is a function that describes how the information about the populations is sampled and converted into the public information made available to the voters, assumed to be a good estimator of  $\mu^e(\omega_{-i})$ . If this information is fully detailed about the state of the system, then it is possible to estimate the vector  $\Delta \equiv [\Delta_1^{t-1}, \Delta_2^{t-1}, \Delta_3^{t-1}]$  by  $\mathbf{m}^t$ , say

$$\mathbf{m}^t = R^c(\Delta^{t-1}) \approx [\Delta_1^{t-1}, \Delta_2^{t-1}, \Delta_3^{t-1}], \quad (11)$$

except for an estimation error.

However, if the public ignores the abstention level in their decision process because, for example, the media only publicize the percentage of voters in favor and against, then

$$\mathbf{m}^t = R^i(\Delta^{t-1}) \approx [\Delta_1^{t-1}, 0, \Delta_3^{t-1}]. \quad (12)$$

This would make the contribution of the abstention level to the individual utility equal to zero.

Another possible scenario occurs when the public is informed of the percentage of votes in favor and against corrected to the participation level of voters  $\Delta_1 + \Delta_2$ . The function  $R$  would then be

$$R^{i*}(\Delta^{t-1}) = \left[ \frac{\Delta_1^{t-1}}{\Delta_1^{t-1} + \Delta_3^{t-1}}, 0, \frac{\Delta_3^{t-1}}{\Delta_1^{t-1} + \Delta_3^{t-1}} \right]. \quad (13)$$

Introducing equation (10) into the definition of the deterministic component  $\bar{V}$  of the individual utility function yields that

$$\bar{V}(\omega_i, \mathbf{m}^t) = \sum_{\alpha=1}^3 g_{\alpha}(\Delta^{t-1}) \delta[\omega_i, \alpha] \quad (14)$$

where  $g_{\alpha} \equiv f_{\alpha} \circ R^1$ .

## 2.5 Formal solution of the utility maximization problem

Combining equations (10) and (9), and using the solution method described in appendix A, produces a relationship between the consecutive fractions of agents at instants  $t-1$  and  $t+1$

$$\Delta_1^{t+1} = \frac{e^{\beta g_1(\Delta^{t-1})}}{e^{\beta g_1(\Delta^{t-1})} + e^{\beta g_2(\Delta^{t-1})} + e^{\beta g_3(\Delta^{t-1})}} \quad (15)$$

$$\Delta_2^{t+1} = \frac{e^{\beta g_2(\Delta^{t-1})}}{e^{\beta g_1(\Delta^{t-1})} + e^{\beta g_2(\Delta^{t-1})} + e^{\beta g_3(\Delta^{t-1})}} \quad (16)$$

$$\Delta_3^{t+1} = \frac{e^{\beta g_3(\Delta^{t-1})}}{e^{\beta g_1(\Delta^{t-1})} + e^{\beta g_2(\Delta^{t-1})} + e^{\beta g_3(\Delta^{t-1})}} \quad (17)$$

which can be written in a formal and compact way as

$$\Delta_{\alpha}^{t+1} = \mathcal{F}_{\alpha}(\Delta^{t-1}), \text{ with } \alpha = 1, 2 \text{ and } 3,$$

where  $\mathcal{F}_{\alpha}$  is defined by the right side of equations (15) to (17).

The Brouwer fixed point theorem (Brouwer, 1911) guarantees that this iteration process, in which the results of newer polls drive voters, converges to a fixed configuration of the system. This final configuration is an attractive fixed point of the discrete map transformation  $\mathcal{F}_{\alpha}$ ,

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<sup>1</sup>The symbol  $\circ$  represents the function composition which is the application of one function (in this case  $f_{\alpha}$ ) to the results of another (in this case  $R$ ).

defined as

$$\Delta_{\alpha}^* = \mathcal{F}_{\alpha}(\Delta^*). \quad (18)$$

The Brouwer theorem also guarantees the existence of at least one fixed point but allows for more than one.

The populations for each voting attitude  $\alpha$  satisfy the condition  $\Delta_1 + \Delta_2 + \Delta_3 = 1$ , which allows writing the aggregate state of the system in terms of the participation level  $p = 1 - \Delta_2$  and the polarization of opinion  $s = \Delta_1 - \Delta_3$ . Equation (18) can be rewritten in terms of  $p$  and  $s$  as

$$s^{t+1} = \mathcal{F}_1(\Delta^{t-1}) - \mathcal{F}_3(\Delta^{t-1}) \quad (19)$$

$$p^{t+1} = 1 - \mathcal{F}_2(\Delta^{t-1}). \quad (20)$$

Equations (19) and (20) can be transformed into

$$s^{t+2} = p^{t+2} \tanh[\chi(s^t, p^t)] \quad (21)$$

$$p^{t+2} = \frac{\cosh[\chi(s^t, p^t)]}{\cosh[\chi(s^t, p^t)] + 1/2 \exp[\nu(s^t, p^t)]} \quad (22)$$

with  $\chi(s, p) \equiv 1/2 [g_1 - g_3]$ ,  $\nu(s, p) \equiv g_2 - 1/2 [g_1 + g_3]$  and  $g_{\alpha} \equiv g_{\alpha}(\mathbf{m}^{t+1})$ . It is convenient to define the polarization of opinion adjusted to the level of participation as  $\sigma^{t+2} \equiv s^{t+2}/p^{t+2}$ .

Then, equations (21) and (22) can be written as

$$\sigma^{t+2} = \tanh[\tilde{\chi}(\sigma^t, p^t)] \quad (23)$$

$$p^{t+2} = \frac{\cosh[\tilde{\chi}(\sigma^t, p^t)]}{\cosh[\tilde{\chi}(\sigma^t, p^t)] + 1/2 \exp[\tilde{\nu}(\sigma^t, p^t)]} \quad (24)$$

where  $\tilde{\chi}(\sigma, p) \equiv \chi(s, p, p)$  and  $\tilde{\nu}(\sigma, p) \equiv \nu(s, p, p)$  should be considered as functions of  $\sigma$  and  $p$ , only. Notice that when the agents cannot abstain and choose only between voting in favor or

against, the level of participation is fixed to 1 (*i.e.*  $p^{t+2} = p^t = 1$ ) and equation (21) reduces to

$$s^{t+2} = \tanh[\chi(s^t, 1)], \quad (25)$$

which coincides with the expression obtained by Brock and Durlauf (2001a, 2001b) for a model with binary choice.

### 3 Results

In this section we characterize the types of fixed points of the system, as expressed in equation (18). This will characterize the outcome of the referendum according to our model. In a first step we consider only the case of herding effect in which information is complete and agents have access to the full results of the polls. In this case,  $R$  is given by the identity matrix as in equation (11). We next consider the case in which information is incomplete and agents ignore the abstention level. In this case  $R$  is given by equation (12). Finally, we address the case in which multiple outcomes are possible in each of the of the information settings described. In this case the usefulness of polls to forecast the referendum outcome is unclear.

#### 3.1 Complete information

We consider the following model for the matrix  $a_{\alpha\beta}$  in equation (8), reflecting the utilities for the different voting attitudes in the case of complete information:

$$a_{\alpha\beta} = \begin{bmatrix} j & q & 0 \\ q & k & r \\ 0 & r & j \end{bmatrix}, \quad (26)$$

where  $j$  is the utility if voters agree to vote in favor or against (since the herding effect should be equally strong in both directions), and  $k$  is the utility if voters agree to abstain. Parameters  $q$  and  $r$  are utilities that reflect the impact of the abstention level in the decision of agents to vote in favor or against, respectively.

The public information resulting from the polls is given by  $m_\beta^t$  as in equation (10), which can be obtained from the actual populations of agents according to  $m_\beta^t = \sum_\gamma R_{\beta\gamma} \Delta_\gamma^{t-1}$ , where  $R$  is the identity matrix. As a result, for complete information the utility function is simply

$$\bar{V}(\omega_i) = \sum_{\alpha=1}^3 \left[ \sum_{\beta=1}^3 a_{\alpha\beta} \Delta_\beta^{t-1} + u_\alpha \right] \delta(\omega_i, \alpha).$$

The outcome of the referendum, expressed as the percentage of the total number of valid votes that were in favor and against, is

$$\begin{aligned} \%favor &= \frac{p+s}{2p} = \frac{1}{2}(1+\sigma) \\ \%against &= \frac{p-s}{2p} = \frac{1}{2}(1-\sigma). \end{aligned}$$

In Figure 3 we present an illustration of the stationary configurations of the system obtained as the overlap of the solution of equation (19) (drawn in red) and equation (20) (drawn in blue) for  $j = 2.0$ ,  $k = 3.0$ ,  $q = 0.0$ , and  $r = 1.0$ . In this situation there are different attitudes toward abstention. Voters in favor are indifferent to abstention ( $q = 0.0$ ), whereas voters against are sensitive to the level of abstention ( $r = 1.0$ ).

**[Insert Figure 3 here]**

From this figure, the fixed points resulting from this dynamic process are  $s = -0.10$  and  $p = 0.32$ , corresponding to 11% of votes in favor versus 21% against and an abstention level of 68%. This outcome translates into 34% of valid votes in favor and 66% against.

### 3.2 Incomplete information

We now consider the same model for the case of incomplete information with the same matrix  $a_{\alpha\beta}$  as above but with

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Following equation (14), this results in a matrix  $\tilde{a}_{\alpha\beta}$

$$\tilde{a}_{\alpha\beta} = \begin{bmatrix} j & 0 & 0 \\ q & 0 & r \\ 0 & 0 & j \end{bmatrix}, \quad (27)$$

which produces the same utility function as a complete information model with an adjusted matrix  $a_{\alpha\beta}$  given by

$$a_{\alpha\beta} = \begin{bmatrix} j & q/2 & 0 \\ q/2 & 0 & r/2 \\ 0 & r/2 & j \end{bmatrix}. \quad (28)$$

Using the participation level  $p = 1 - \Delta_2$  and the polarization of opinion  $s = \Delta_1 - \Delta_3$ , we present an illustration of the stationary configurations of the system obtained for the same values of the parameters  $j = 2.0$ ,  $k = 3.0$ ,  $q = 0.0$ , and  $r = 1.0$  as those used in the example above of complete information.

Under these conditions we have  $s = 0.23$  and  $p = 0.82$ . As a result, the outcome translates into 64% in favor and 36% against, with a level of abstention of 18%. Comparing the results in Figures 3 and 4, we notice that under incomplete information there is a radical reduction of the abstention level (from 68% to 18%) as compared to the case of complete information, together with a change of the referendum result, from a rejection to an approval of the issue at stake.

**[Insert Figure 4 here]**

### 3.3 Multiple outcomes

In the examples above we have considered that there is only one stationary configuration of the system, i.e. a single pair  $(s,p)$  such that both equations (19) and (20) are simultaneously

satisfied. We now consider the case in which the parameters of the model are such that more of one pair can simultaneously satisfy both equations. This is illustrated in Figure 5, where on the left side we have cases of one single solution and on the right side the chosen parameters lead to three possible solutions.

**[Insert Figure 5 here]**

There are two features that we analyze in the Figure 5 graphs: i) the break of symmetry (visible in the change from the first to the second line); ii) the increase in the number of solutions (visible in the change from the left to the right side).

In the first feature, the two figures in the top line are perfectly symmetrical. The change to the bottom line breaks the symmetry of the solutions. From the point of view of the parameters,  $q$  is different from  $r$  in the bottom line, reflecting different predispositions toward abstention between the two different groups. This means that one of the stands in the referendum provides a stronger motivation to vote - or to fight abstention - than the other stand. If we take the left column, we can see that this asymmetry favors one of the outcomes, namely the position of the group that is more mobilized to vote.

While the cases on the left side are similar to the examples given in the earlier sections, the multiple solution cases deserve some additional attention. We notice that in the transition from the left to the right column the value of  $j$  increases. This indicates that the social interaction - and thus the peer pressure - within each stand is stronger in the right side, making it more difficult for people to change opinion after repeated interactions. In other words, a high value of  $j$  implies that groups that succeed in attracting undecided voters are more likely to keep them, as compared to the case where  $j$  is low. This leads to a stable solution of the system, i.e. a solution in which most voters are strongly committed and the probability of changing their minds is very low. For that reason, such solutions are termed *stable*. The fastest growing group in this process will eventually gain the advantage in the referendum process. As we cannot ex-ante identify which group will achieve this leading position, the model yields both of the stable outcomes that are possible: one in which the position in favor wins and another in which the position in favor loses.

We also notice that in the cases corresponding to the right column of Figure 5 there is a third intermediate solution yielding a close match between the number of voters against and in favor. This basically corresponds to a tie or, more precisely, a state of indecision in which the system has not achieved a stable solution. In such a situation, a few people changing opinion may easily swing the solution to either one of the stable solutions. Even though this solution satisfies equations (19) and (20), and is a fixed point of the dynamics, it is *unstable* and very difficult to identify in real terms (see Figure 6).

[Insert Figure 6 here]

## 4 Calibration of the model

We now turn to an essential contribution of this approach. The issue we address here is to understand how the data gained from polls can be used to tell us something about the system that, in turn, will allow us to forecast the final result of the referendum with a satisfactory degree of accuracy. In other words, we would like to know the extent to which the sequence of poll data will allow us to correctly calibrate the model, determining the values of the parameters including the matrix  $a$  of interaction between voters and their individual preferences  $h$ .

Applying this procedure to the particular case of the abortion referendum in Portugal (1998), we will consider a matrix

$$a_{\alpha\beta} = \begin{bmatrix} j_+ & q & 0 \\ q & k & r \\ 0 & r & j_- \end{bmatrix}$$

where we use two different parameters  $j$ :  $j_+$  for the social pressure among voters in favor, and  $j_-$  for the social pressure among voters against. For simplicity, we assume a generating function  $G(y_1, y_2, \dots)$  of the form

$$G(y_1, y_2, y_3) = y_1 + y_2 + y_3.$$

The data for the polls publicized for the 1998 referendum were obtained from Magalhães

(1998), being careful not to include the results of the final referendum voting. We used non-linear minimum least square fitting to obtain

$$\begin{aligned}\beta \times a_{\alpha\beta} &= 5.62 \times \begin{bmatrix} 8.28 \times 10^{-2} & 2.34 \times 10^{-1} & 0 \\ 2.34 \times 10^{-1} & 4.58 \times 10^{-7} & -6.65 \times 10^{-1} \\ 0 & -6.65 \times 10^{-1} & 2.07 \times 10^{-8} \end{bmatrix} \\ \beta \times h_{\alpha} &= 5.62 \times \begin{bmatrix} 8.81 \times 10^{-3} \\ -2.28 \times 10^{-1} \\ 2.15 \times 10^{-2} \end{bmatrix}\end{aligned}$$

We notice several interesting things in this matrix.

First, the value of  $k$  is zero, meaning that the system behaves as if information were incomplete, in the sense that no emphasis was given in the media to the (declared) abstention level.

Second, the value of  $q$  is positive while the value of  $r$  is negative. This shows that voters against abortion were much more mobilized to vote than sympathizers of abortion legalization. This difference in mobilization actually could explain that most of the few final voters were against abortion, contradicting the polls. This is consistent with much of the subsequent analysis in the press at that time, which sought to explain the surprising result of the referendum.

We should also analyze the diagonal elements  $j_+$  and  $j_-$ . Notice that the former is positive and the latter is zero for all practical purposes. Recalling that these numbers express the intra-group social pressure, we might expect that this simply reflects the fact that the arguments conditioning the attitudes of those in favor were strongly induced by social pressure. For those against, the arguments were clearly more of an idiosyncratic nature, based on personal convictions that did not require reinforcement of social pressure.

As described in the introduction, the referendum was repeated in 2007. The fitting of the results of the polls preceding the second referendum (Magalhães, 2007), yielded

$$\beta \times a_{\alpha\beta} = 1.58 \times \begin{bmatrix} 4.97 \times 10^{-1} & -3.56 \times 10^{-1} & 0 \\ -3.56 \times 10^{-1} & 1.44 \times 10^{-8} & -5.00 \times 10^{-1} \\ 0 & -5.00 \times 10^{-1} & 3.37 \times 10^{-8} \end{bmatrix}.$$

$$\beta \times h_{\alpha} = 1.58 \times \begin{bmatrix} 3.08 \times 10^{-2} \\ -2.83 \times 10^{-1} \\ 1.05 \times 10^{-2} \end{bmatrix}$$

In light of the above analysis, it is clear what happened in this second referendum. In qualitative terms the only change in the matrix is that the value of  $q$  is now negative. This simply means that the group in favor changed their attitude toward abstention, becoming much more active in mobilizing people to vote. This is consistent with the explanation advanced by the analysts following the first referendum, that the outcome contradicting the polls resulted from a low mobilization of the majority of voters, namely those in favor.

## 5 Static and dynamic analysis

We now provide some comparative statics of the solution of the model matrix, say

$$\beta \times a_{\alpha\beta} = \begin{bmatrix} j_+ & q & 0 \\ q & 0 & r \\ 0 & r & 0 \end{bmatrix}. \quad (29)$$

Notice that in this matrix the values of  $k$  and  $j_-$  have been set to zero in accordance with the results of the calibration described in the previous section. For this situation the functions in equations (23) and (24) become

$$\tilde{\chi}(\sigma, p) = \frac{1}{4}j_+p(\sigma + 1) + \frac{1}{2}\Gamma(1 - p) \quad (30)$$

$$\tilde{\nu}(\sigma, p) = \frac{1}{4}j_+p(\sigma + 1) + \frac{1}{2}\Lambda(1 - p) \quad (31)$$

where  $\Gamma \equiv (q - r)/2$  and  $\Lambda \equiv (q + r)/2$  denote respectively the asymmetry and average in the cross-herding power toward abstention between voters against and in favor.

Consider first how small changes of the polarization of opinion  $\delta\sigma^t$  and of the voter participation  $\delta p^t$  at instant  $t$  produce variation of the voters' decision at instant  $t + 2$ , namely  $\delta\sigma^{t+2}$  and  $\delta p^{t+2}$ :

$$\delta\sigma^{t+2} = \left[1 - (\sigma^{t+2})^2\right] \frac{1}{4} j_+ p^t \delta\sigma^t + \left[1 - (\sigma^{t+2})^2\right] \left[ \frac{1}{4} j_+ (\sigma^t + 1) - \Gamma \right] \delta p^t \quad (32)$$

$$\begin{aligned} \delta p^{t+2} = & p^{t+2} \left[1 - p^{t+2}\right] \left[ \frac{1}{4} j_+ (\sigma^{t+2} + 1) - \Gamma \right] p^t \delta\sigma^t + \\ & + \frac{1}{4} p^{t+2} \left[1 - p^{t+2}\right] \left[ j_+ (\sigma^{t+2} + 1)(\sigma^t + 1) - \Gamma(\sigma^{t+2} + \sigma^t) - 2\Lambda \right] \delta p^t \end{aligned} \quad (33)$$

In the neighborhood of each fixed point, the variation of the results of polls becomes small, and it then follows that  $\delta\sigma^{t+2} \approx \sigma^{t+2} - \sigma^t$ ,  $\delta p^{t+2} \approx p^{t+2} - p^t$ ,  $\sigma \approx \sigma^{t+2} \approx \sigma^t$  and  $p \approx p^{t+2} \approx p^t$ .

The feedback effect for  $\sigma$  between instants  $t$  and  $t + 2$  is given by

$$\delta\sigma^{t+2} = [1 - \sigma^2] \frac{1}{4} j_+ p \delta\sigma^t \quad (34)$$

and is always positive ( $\delta\sigma^t > 0$  implies that  $\delta\sigma^{t+2} > 0$ ), indicating a herding effect that reinforces the results of the previous polls. On the other hand, the feedback effect for  $p$  between instants  $t$  and  $t + 2$  is given by

$$\delta p^{t+2} = \frac{1}{4} p [1 - p] [j_+ (\sigma + 1)^2 - 2\Gamma\sigma - 2\Lambda] \delta p^t. \quad (35)$$

The analysis of this feedback does not have a well-defined signal, being positive for  $j_+ (\sigma + 1)^2 - 2\Gamma\sigma - 2\Lambda > 0$  and negative for  $j_+ (\sigma + 1)^2 - 2\Gamma\sigma - 2\Lambda < 0$ . To explain the meaning of the change of sign of  $\delta p^{t+2}$  assume for simplicity that  $\Gamma$  is positive and large ( $\Gamma/j_+ > 2$ ) and  $\Lambda = 0$ . In other words,  $\Gamma > 0$  means that the voters in favor have a strong cross-herding inducement toward abstention. As a result, when  $\sigma$  is negative or positive but small and voters in favor are not likely to win, or there is a close match in the results, they then tend to abstain less and support their stand resulting in a positive feedback ( $\delta p^{t+2}/\delta p^t > 0$ ). However, as  $\sigma$  gets

close to 1, and it is foreseen that the stand in favor is going to win with a large majority, some of these voters will tend to abstain, resulting in a negative feedback ( $\delta p^{t+2}/\delta p^t < 0$ ). These negative feedbacks are very interesting dynamically because they entail that the variations of participation will change sign in two consecutive polls, meaning that the participation level fluctuates (decreasing after increasing and vice-versa). This latter result can be interpreted as a reaction to a decrease in the perceived pivotal power of voters both in favor and against. For  $0 < \Gamma/j_+ < 2$  the effect of cross-herding toward abstention of the voters in favor is not strong enough to switch the sign of the feedback for large  $\sigma$  but only a decrease amplitude of  $\delta p^{t+2}/\delta p^t$ . For negative  $\Gamma/j_+$  the cross-herding power is stronger for voters against and now occurs the symmetrical case, where for large negative  $\sigma$  the participation fluctuates and for weakly and positive  $\sigma$  there is always a positive feedback.

The impact of the level of participation in the polarization or gap  $\sigma$  between voting attitudes  $w = 1$  and  $w = 3$  is obtained from

$$\delta \sigma^{t+2} = [1 - \sigma^2] \left[ \frac{1}{4} j_+ (\sigma + 1) - \Gamma \right] \delta p^t. \quad (36)$$

In the absence of asymmetry toward abstention between voter in favor and against ( $\Gamma = 0$ ), increases in participation level  $p$  tend to amplify the gap  $g$  and therefore the trend of the polls at  $t$  are reinforced at  $t + 2$ . When an asymmetry is introduced ( $\Gamma \neq 0$ ) this amplification can be either increased ( $\Gamma < 0$ ) or decreased ( $\Gamma > 0$ ), and even inverted ( $\Gamma > \frac{1}{4} j_+ (\sigma + 1)$ ).

The level of participation  $p$  is affected by the gap  $g$  according to

$$\delta p^{t+2} = p^2 (1 - p) \left[ \frac{1}{4} j_+ (\sigma + 1) - \Gamma \right] p \delta \sigma^t. \quad (37)$$

Notice that the sign of this feedback is identical to the previous expressions. As before, in the absence of an asymmetric cross-herding ( $\Gamma = 0$ ), an increase in voter polarization toward the voting option exhibiting pure herding ( $w = 1$ ) promotes voter participation. However, an inversion of the polarization ( $\sigma < 0$ ) tends to decrease voter participation because  $w = 2$  presents no herding effect. Again, the effect of  $\Gamma \neq 0$  is to distort the pure herding effect on

the participation level.

## 6 Conclusions

In this paper we propose an explanation for differences between the polls projections and the final outcome of referenda. Our model implies that there is a dynamic feedback between the results of polls and individual voting attitudes. This process can be affected by the way results of polls are presented to the public in the media and, ultimately, by the way they are perceived by voters. Dissonances between individual perception and aggregate voting trends can be amplified by the feedback process, resulting in a strong discrepancy between the results of polls and the actual voting outcome.

One such dissonance can occur when individuals within each of the voting groups have different predispositions toward abstention. This bias may not be explicit in the results of the polls as divulged by the media, which are more focused on the number of voting intentions in favor and against. Individuals are then not able to correctly anticipate the behavior of others in elections, and adjust their voting attitude accordingly. This can favor smaller voting groups with larger aversion toward abstention and eventually switch the outcome of the referendum from the polls' predictions.

From the dynamic point of view, we have shown that this model always has at least one attractive fixed point at which the number of individuals adopting each voting attitude is stable. The results of successive polls drive the aggregate voting attitudes and asymptotically converge to the attractive fixed point.

Under certain conditions, the model can have more than one fixed point, and it is then impossible to determine ex-ante the fixed point towards which the aggregate voting attitudes will converge. This can further frustrate polls in forecasting the outcome of referenda.

To illustrate the application of real data to this model, we considered the case of the two abortion referenda in Portugal that occurred in 1998 and 2007. In 1998 the stand against the proposition in the referendum had the highest numbers of votes despite the fact that all the polls

had pointed strongly in the opposite direction. At the time, many political analysts justified this discrepancy with a stronger predisposition to vote among individuals against the issue. The debate following the results of this referendum increased the social pressure for voters in favor not to abstain in the second referendum. Also, there was a more intense discussion of the impact of level of abstention in the voting results. In the voting results of 2007 the stand in favor had the largest number of votes, thus confirming the original polls.

Using the results of the polls in the run-up to each referendum, it is possible to calibrate the model incorporating cross-herding effects. For the parameters characterizing such effects, it was found that their values were consistent with the above-mentioned interpretation of the results. In particular, the calibration of the model provides evidence that voters in favor reduced their predisposition toward abstention following the 1998 results, leading to greater participation in the 2007 referendum.

Our model is sufficiently general to encompass the pivotal effect, i.e., the explanation that an individual may be led to vote in spite of the costs to do so, because there is a chance that his or her vote may be decisive. Since it has been argued in the literature that this effect is negligible in both theoretical and empirical terms, we chose not to include this (rather simple) extension herein. However, our findings will be provided to the interested reader upon request.

With suitable adaptations the model presented here can be applied to other types of voting processes, namely elections. As discussed above, there are two main differences. The most obvious change would be that the number of voting choices would be greater than two. A second more subtle issue is that the choice process in regular elections is a repeated game recurring every four years - and not the one-shot election that a referendum usually is. In other words, on the basis of the choice in a regular election (local, state, or national) there are factors such as the past performance of the incumbent to be taken into account, leading to a different inter-temporal utility structure that we did not touch upon. This model stresses the dynamic and collective nature of the voting decision process and the implications of the completeness of information available to the public in the outcome of that decision. These factors allow for feedback loops between voters that can amplify trends and produce results that can be quite

different from the poll forecast.

## Figures

Institution responsible for poll	Date	Voting intentions (%)			Valid votes (%)	
		yes	no	undecided	yes	no
Metris	4/8/1998	48.7	41.3	9.9	54	46
SIC/Visão	5/1/1998	63.1	35.3	1.6	64	36
Católica	5/5/1998	60.9	30.3	8.8	67	33
Metris	5/20/1998	46.9	43.9	9.1	52	48
SIC/Visão	5/21/1998	60.9	36.7	2.5	62	38
Moderna	5/24/1998	55.9	33.1	11.4	63	37
Euroexpansão	5/27/1998	81.0	13.0	6.0	86	14
SIC/Visão	6/9/1998	58.1	40.0	1.9	59	41
Moderna	6/14/1998	54.4	35.5	10.1	61	39
Metris	6/17/1998	44.4	41.6	13.4	52	48
Católica	6/19/1998				53.2	46.8

Final results (%)	
yes	49.1
no	50.9
other	1.6
voting participation	31.9
Abstention	68.1

Figure 1: Results of the opinion polls leading up to the 1998 national referendum on abortion in Portugal. Source: <http://margensdeerro.blogspot.com>

Institution responsible for poll	Date	Voting intentions (%)			Valid votes (%)	
		yes	no	undecided	yes	no
Aximage	4/20/2006	47.9	39.9	12.2	55	45
Católica	15/10/2006	63.0	25.0	12.0	72	28
Eurosondagem	17/10/2006	46.0	39.0	16.0	54	46
Marktest	21/10/2006	63.0	27.0	10.0	70	30
Intercampus	24/10/2006	66.9	27.6	5.4	71	29
Marktest	15/11/2006	61.0	30.0	9.0	67	33
Aximage	7/12/2006	64.1	27.3	8.6	70	30
Aximage	20/12/2006	61.0	26.7	12.3	70	30
Intercampus	9/1/2007	60.0	29.0	11.0	67	33
Aximage	9/1/2007	57.0	34.8	8.1	62	38
Eurosondagem	16/01/2007	43.0	36.0	22.0	54	46
Aximage	18/1/2007	55.6	29.8	14.6	65	35
Marktest	19/1/2007	54.0	33.0	13.0	62	38
Católica	22/1/2007	48.0	35.0	17.0	58	42
IPOM	24/1/2007	45.6	37.2	17.2	55	45
Aximage	2/2/2007	51.3	43.7	5.0	54	46
Católica	4/2/2012	49.0	34.0	17.0	59	41
TNS	5/2/2007	52.8	37.1	10.1	59	41
Eurosondagem	5/2/2007	46.5	41.0	12.5	53	47
Aximage	7/2/2007	52.6	41.5	5.9	56	44

Final results (%)	
yes	49.1
no	50.9
other	1.6
voting participation	31.9
Abstention	68.1

Figure 2: Results of the opinion polls leading up to the 2007 national referendum on abortion in Portugal. Source: <http://margensdeerro.blogspot.com>

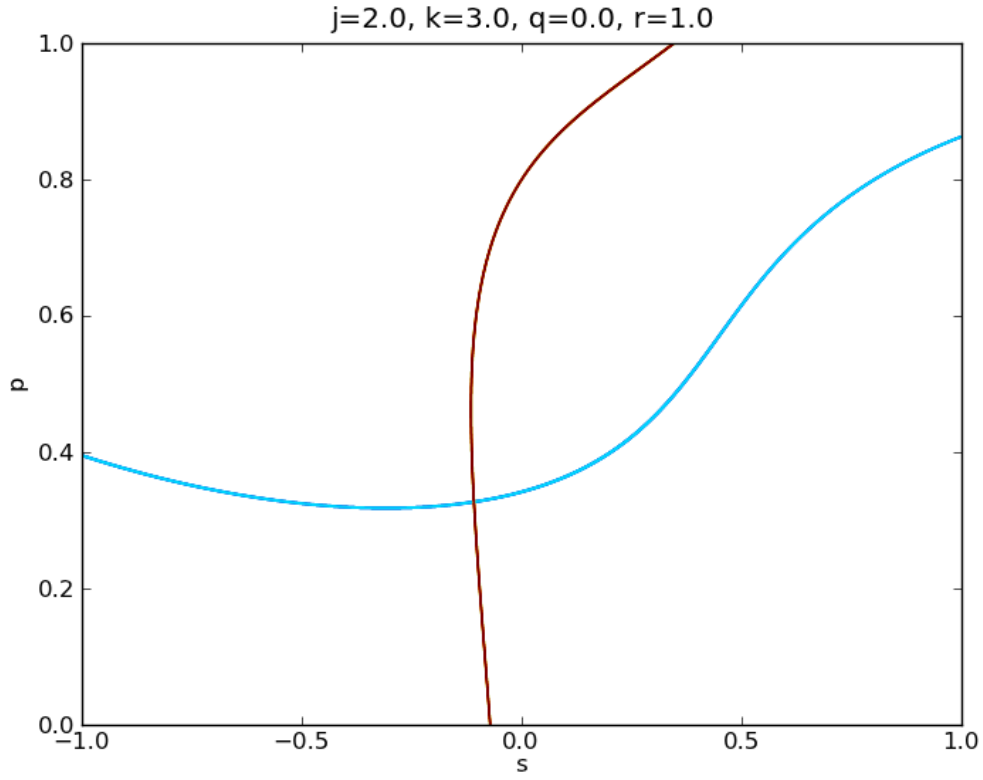


Figure 3: Graphic determination of the stationary configuration of a social system of voters as solutions of equations (19) (drawn in red) and (20) (drawn in blue), for parameters  $j = 2.0$ ,  $k = 3.0$ ,  $q = 0.0$ ,  $r = 1.0$ , and  $h \equiv [h_1, h_2, h_3] = [0.2, 0., 0.0]$ .

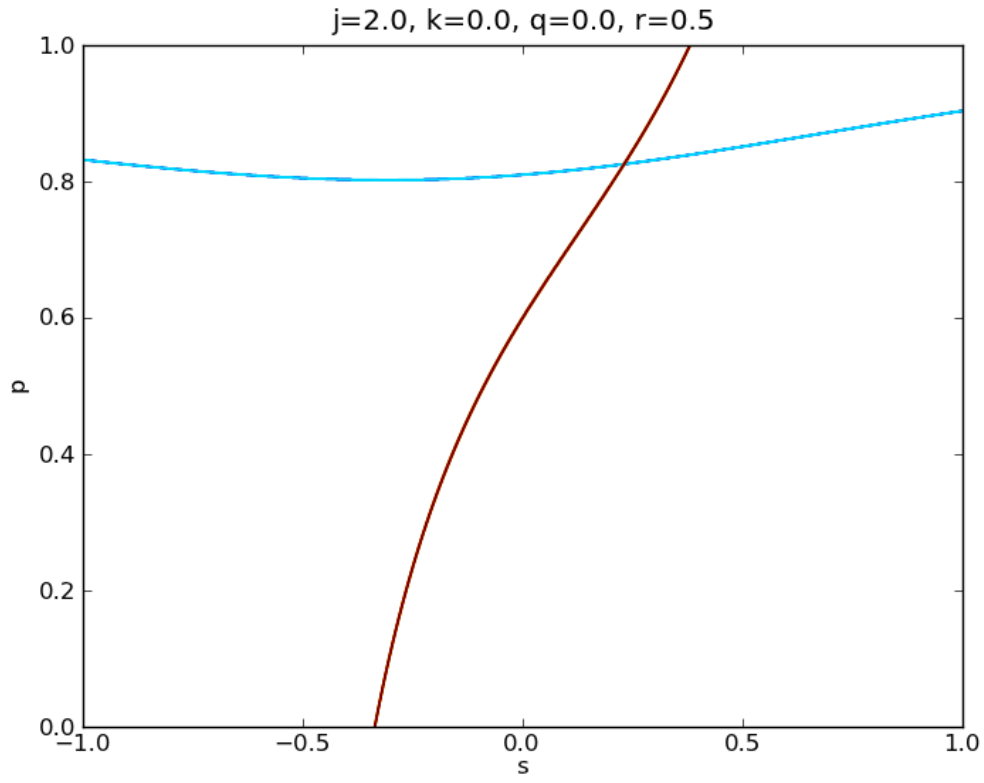


Figure 4: Graphical determination of the stationary configuration of a social system of voters as solutions of equations (19) (drawn in red) and (20) (drawn in blue), for parameters  $j = 2.0$ ,  $k = 3.0$ ,  $q = 0.0$ ,  $r = 1.0$ , and  $h \equiv [h_1, h_2, h_3] = [0.2, 0., 0.0]$ .

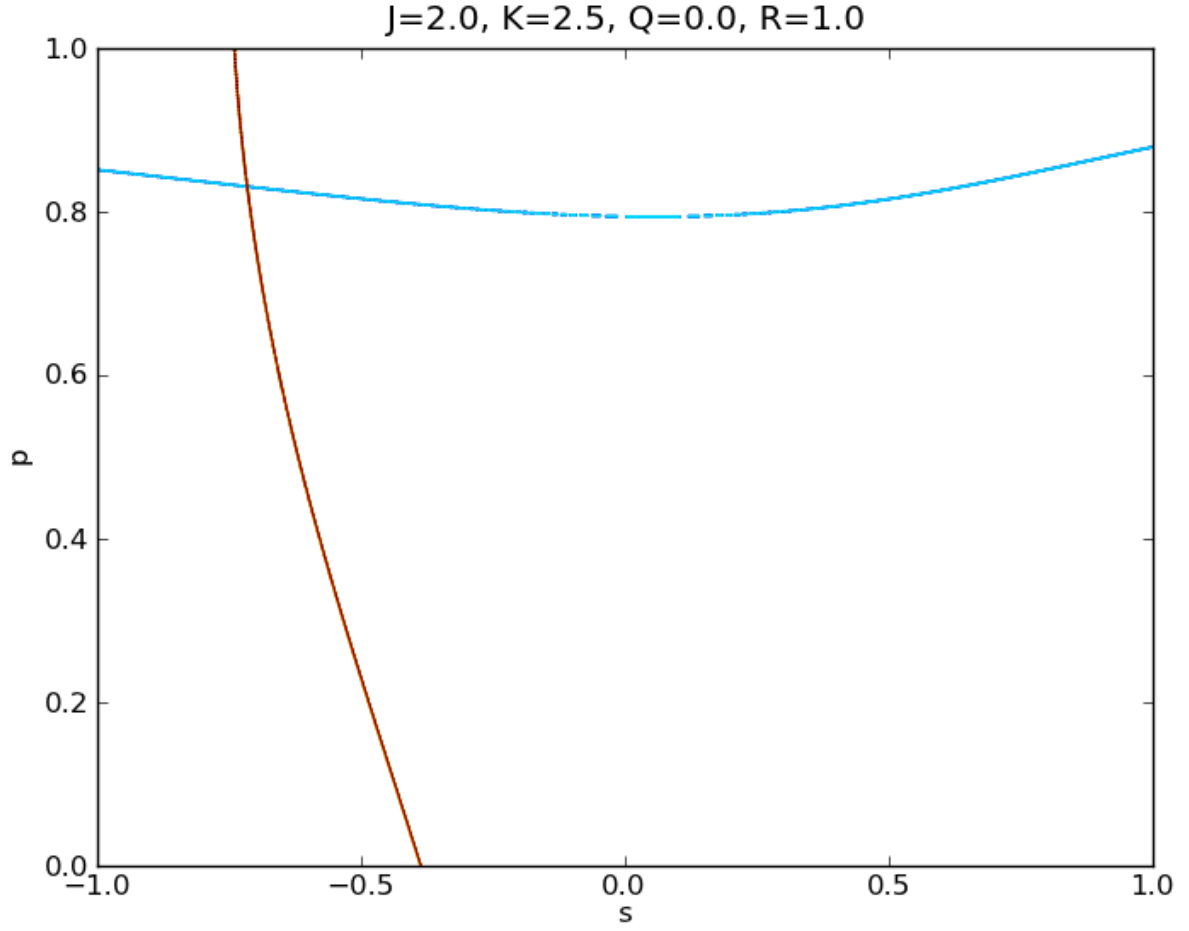


Figure 5: Graphic determination of the stationary configuration of a social system of voters as solutions of equations (19) (drawn in red) and (20) (drawn in blue), for different set of parameters. It is shown that for several primitives, the system can present one (on the left) or multiple (on the right) stationary points. In the case of multiple stationary points, they are not all necessarily attractive. In the cases presented only the two most outer points are attractive.

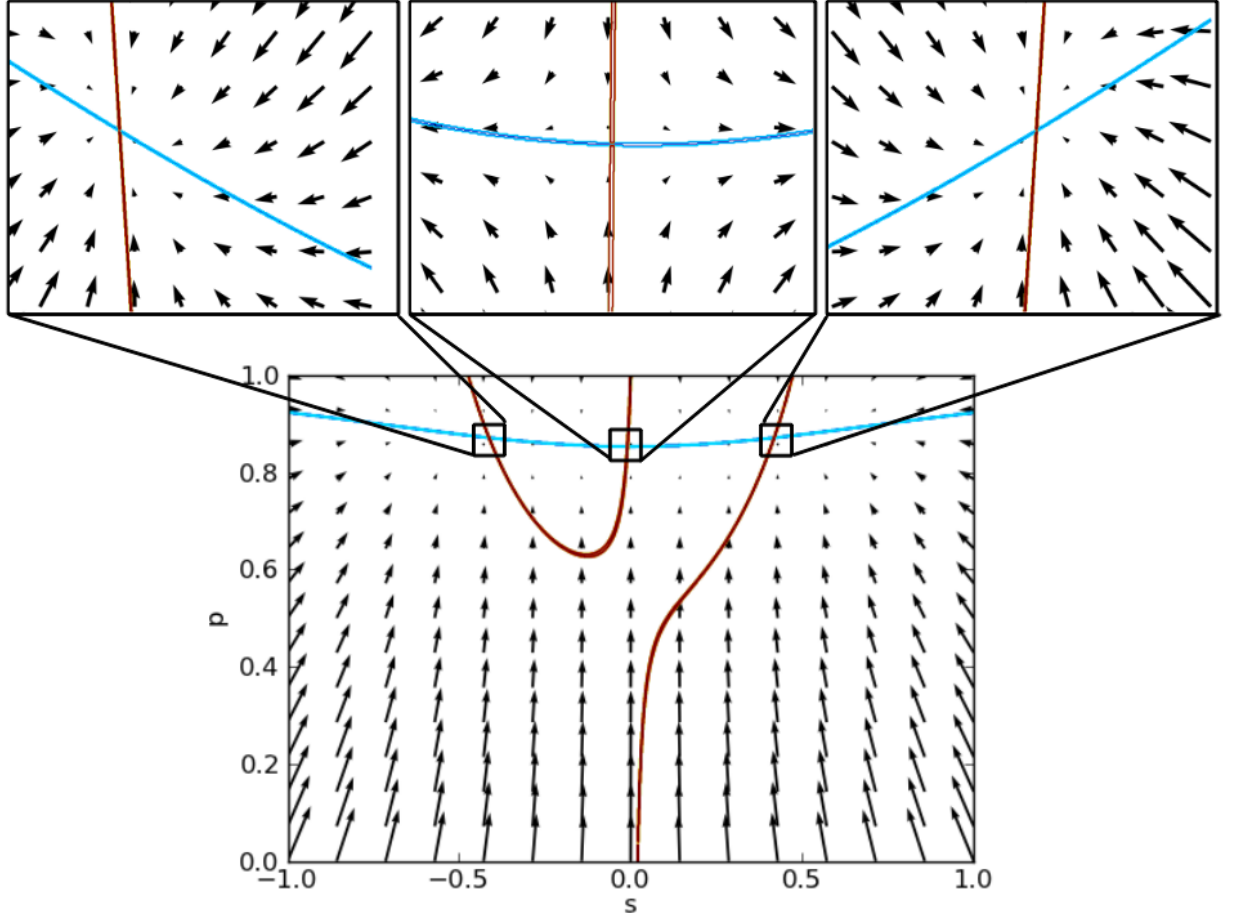


Figure 6: Graphic determination of the stationary configuration of a social syste of voters as solutions of equations (19) (drawn in red) and (20) (drawn in blue), in a case with multiple stationary points. The arrows connect two consecutive configurations according to the dynamics of the system. On top we present amplifications of the graph in the proximity of each of the stationary points. It is shown that the two outer stationary points are stable while the central one is unstable.

## Appendix A

McFadden (1978, 1981) developed a method for solving the maximization problem

$$\omega_i^{*t+1} = \underset{\omega_i=1,2,3}{argmax} V_i^{t+1}(\omega_i, \mathbf{m}^t) \quad (38)$$

in a wide class of situations. His approach begins by considering that the random term of the individual utility function  $\epsilon_{i,\alpha}$  follows a frequency distribution obtained from generating function  $G \equiv G(y_1, y_2, \dots)$ . The function  $G$  is assumed to be a non-negative homogenous function of degree  $d > 0$  defined on the orthant  $y_\alpha \geq 0$ , which diverges without bound. The  $k$ th cross partial derivatives of  $G$  are non-negative when  $k$  is odd and non-positive when  $k$  is even. The valid multivariate distribution function generated by  $G$  is

$$F_G = \exp[-G(y_1, y_2, \dots)]. \quad (39)$$

McFadden (1978, 1981) showed that if  $\epsilon_{i,\alpha}$  jointly follows this type of multivariate generalized extreme value with generating function  $G$ , the probability  $P(\omega_i^{*t+1} = \omega_i)$  can be derived in closed form as

$$\begin{aligned} P(\omega_i = 1) &= \frac{y_1 G_1(y_1, y_2, \dots)}{G(y_1, y_2, \dots)}, \\ P(\omega_i = 2) &= \frac{y_2 G_2(y_1, y_2, \dots)}{G(y_1, y_2, \dots)}, \\ P(\omega_i = 3) &= \frac{y_3 G_3(y_1, y_2, \dots)}{G(y_1, y_2, \dots)}, \end{aligned}$$

with  $y_\alpha = \exp[\beta f_\alpha^t]$ ,  $G_\alpha = \partial G / \partial y_\alpha$ , and  $\beta$  some scalar parameter. A more recent formulation of these results can be found in Misra (2005).

It is convenient to distinguish between individual quantities referring to a particular agent  $i$ , say  $X_i \equiv X_i(\omega_i)$ , which depend on the state of the agent alone, and global quantities  $X \equiv \sum_{i=1}^N p_i X_i$ , which aggregate individual quantities of all the agents according to some weights  $p_i$ .

The average value of  $X_i$  over the probability distribution  $P(\omega_i)$  for an individual  $i$  is defined as the quantity

$$\langle X_i \rangle = \sum_{\alpha=1}^3 X_i(\alpha) P(\alpha),$$

and can be calculated using an alternative expression obtained by construction as

$$\langle X_i \rangle = \left. \frac{\partial \ln G}{\partial Y} \right|_{Y=0} = \sum_{\alpha=1}^3 X_i(\alpha) e^{X_i(\alpha)Y} \frac{y_\alpha G_\alpha}{G} \Big|_{Y=0} = \sum_{\alpha=1}^3 X_i(\alpha) \frac{y_\alpha G_\alpha}{G} = \sum_{\alpha=1}^3 X_i(\alpha) P(\alpha)$$

Following the same approach, the variance of  $X_i$  is

$$\text{var}(X_i) \equiv \langle X_i^2 \rangle - \langle X_i \rangle^2 = \left. \frac{\partial^2 \ln G}{\partial Y^2} \right|_{Y=0}.$$

Also, the mean value of global quantities  $X$  can be computed by

$$\langle X \rangle = \left. \frac{\partial \ln G}{\partial Y} \right|_{Y=0} = \sum_{i=1}^N p_i \left[ \sum_{\alpha=1}^3 X_i(\alpha) \frac{y_\alpha G_\alpha}{G} \right].$$

In particular, the value of  $\langle \Delta \rangle$  can be calculated using the fact that  $\Delta_\alpha^{t+1} = N^{-1} \sum_{i=1}^N \delta(\omega_i, \alpha)$  (i.e. with  $p_i = 1/N$  and  $X_i = \delta(\omega_i, \alpha)$ ). Then

$$\langle \Delta_\alpha^{t+1} \rangle = \sum_{i=1}^N p_i \left[ \sum_{\beta=1}^3 X_i(\alpha) \frac{y_\alpha G_\alpha}{G} \right] = \sum_{i=1}^N \left[ \sum_{\alpha=1}^3 \delta[\alpha, \omega_i] \frac{y_\alpha G_\alpha}{G} \right] = \frac{y_\omega G_\omega}{G}. \quad (40)$$

In the right side of equation (40) there is an implicit functional dependence of  $\langle \Delta_w^{t+1} \rangle$  in the results of polls obtained at  $t$  via  $y_\alpha = \exp[\beta f_\alpha(\mathbf{m}^t)] = \exp[\beta g_\alpha(\Delta^{t-1})]$ .

This dependence can be expressed in a formal way as

$$\langle \Delta_\alpha^{t+1} \rangle = \mathcal{F}_\alpha(\mathbf{m}^t) \quad (41)$$

with  $\mathcal{F}_\alpha(\mathbf{m}^t) \equiv y_\alpha G_\alpha / G$ . To simplify notation, the average brackets  $\langle \rangle$  are omitted elsewhere in this article.

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