Optimization: A Journal of Mathematical Programming and Operations Research
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gopt20

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M. Álvarez-Mozos[a], F. Ferreira[b], J.M. Alonso-Meijide[c] & A.A. Pinto[d]

[a] Department of Statistics and Operations Research, Faculty of Mathematics, University of Santiago de Compostela, La Coruña, Spain.

[b] ESEIG, Polytechnic Institute of Porto, Porto, Portugal.

[c] Department of Statistics and Operations Research, Faculty of Sciences of Lugo, University of Santiago de Compostela, La Coruña, Spain.

[d] LIAAD-INESC TEC, Departament of Mathematics, Faculty of Sciences, University of Porto, Porto, Portugal.

Published online: 22 Feb 2013.

To cite this article: M. Álvarez-Mozos, F. Ferreira, J.M. Alonso-Meijide & A.A. Pinto (2015) Characterizations of power indices based on null player free winning coalitions, Optimization: A Journal of Mathematical Programming and Operations Research, 64:3, 675-686, DOI: 10.1080/02331934.2012.756878

To link to this article: http://dx.doi.org/10.1080/02331934.2012.756878

Optimization, 2015
Vol. 64, No. 3, 675–686, http://dx.doi.org/10.1080/02331934.2012.756878

Characterizations of power indices based on null player free winning coalitions

M. Álvarez-Mozos[a]*, F. Ferreira[b], J.M. Alonso-Meijide[c] and A.A. Pinto[d]

[a] *Department of Statistics and Operations Research, Faculty of Mathematics, University of Santiago de Compostela, La Coruña, Spain;* [b] *ESEIG, Polytechnic Institute of Porto, Porto, Portugal;* [c] *Department of Statistics and Operations Research, Faculty of Sciences of Lugo, University of Santiago de Compostela, La Coruña, Spain;* [d] *LIAAD-INESC TEC, Departament of Mathematics, Faculty of Sciences, University of Porto, Porto, Portugal*

(*Received 9 May 2012; final version received 3 December 2012*)

In this paper, we characterize two power indices introduced in [1] using two different modifications of the monotonicity property first stated by [2]. The sets of properties are easily comparable among them and with previous characterizations of other power indices.

Keywords: simple game; power index; characterization

1. Introduction

In the last decades, the measurement of power in decision-making bodies such as the European Union Council of Ministers or the International Monetary Fund has been a main topic in political sciences and many work has been done in order to attain an appropriate measure. However, there is still a debate even on the definition of power. Most of the times, the power is understood as the ability of an agent to influence the outcome. But, even when the definition of power is agreed, the choice of an appropiate rule to represent it is still an open question.

Among the most studied power indices in the literature, one can find the Shapley–Shubik index [3], the Banzhaf index [4], the Deegan–Packel index [5] and the Public Good index [6]. All the above power indices are evaluations of an agent's relative significance to each of the coalitions that might be formed. In this work, we will first of all review some of the main results related to these four power indices. Some of the aforementioned power indices restrict their attention to some kinds of coalitions that are particularly important. Indeed, the Deegan–Packel and Public Good indices only take into account the so-called minimal winning coalitions. A winning coalition, that is, a group of agents that can pass a bill on their own, is a minimal winning coalition when the removal of any of its members would prevent the coalition from passing the bill. More recently, other interesting power indices have been introduced. The Public help index [7] is based on the set of all winning coalitions, more precisely, the power of each agent is proportional to the number of winning coalitions in which he participates. The Shift power index [8] considers only a subset of the

*Corresponding author. Email: mikel.alvarez@ub.edu

minimal winning coalitions, the so-called minimal winning coalitions without any surplus, in a sense, these coalitions are the most efficient minimal winning coalitions. In this paper, we take up again the two power indices introduced in [1], namely f^{np} and g^{np}, and study their properties. These power indices are also based on a particularly important set of coalitions, specifically on the winning coalitions that do not contain null players. A null player is a player whose participation in any coalition does not change the situation, i.e. the coalition continues being either winning or loosing. Such set of coalitions contains the set of minimal winning coalitions and is contained in the set of winning coalitions. A first consequence of this fact is that f^{np} and g^{np} do not consider minimal winning coalitions as the only source of power. This is the case in many real situations for instance, many times the adopted decisions are more stable the greater the winning coalition supporting it is. Hence, the information set on which the new power indices are based is wider than the information set on which the Deegan–Packel and Public Good indices are based.

The modelling of decision-making bodies and voting procedures has been tackled using simple games. The axiomatic characterization of power indices is a main topic in the field for at least two reasons. First, characterizing a rule by means of a set of properties may be more appealing than just giving its explicit definition. Second, deciding on whether to use a rule or another in a particular situation may be done more easily taking into account the properties that each rule satisfies. In fact, many power indices have shown to have different sets of properties that determine them uniquely. In this document, we present parallel characterizations of the two power indices introduced in [1] in line with the characterization of the Deegan–Packel power index by [9] and the Public Good index by [10]. In this way, the comparison among these four power indices is much easier since the characterizations only differ in one property. Moreover, the property in each of the characterizations is a type of monotonicity in the sense that it describes the way in which the payoff of an agent changes when his position in the situation is improved.

The rest of the paper is organized as follows. In Section 2, we announce notation and present some preliminary definitions and results such as the definitions and characterizations of the Shapley–Shubik and Banzhaf power indices. In Section 3, the Deegan–Packel and Public Good power indices are presented together with a pair of characterizations of each one of them. In Section 4, the main results of the paper are presented, that is, the new power indices f^{np} and g^{np} are characterized by means of similar properties to the ones used in the characterizations presented in Section 3. Finally, Section 5 discusses some concluding remarks.

2. Preliminaries

A *cooperative transferable utility game (just game from now on)* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the (finite) set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game, which satisfies $v(\emptyset) = 0$. In general, we interpret $v(S)$ as the benefit that S can obtain by its own, i.e. independent to the decisions of players in $N \setminus S$. To avoid cumbersome notation, braces will be omitted whenever it does not lead to confusion; for example, we will write $v(S \cup i)$ or $v(S \setminus i)$ instead of $v(S \cup \{i\})$ or $v(S \setminus \{i\})$. A player $i \in N$ is a *null player* in a game (N, v) when his marginal contribution to every coalition is zero, i.e. when for every $S \subseteq N \setminus i$, $v(S \cup i) - v(S) = 0$. Two players $i, j \in N$ are *symmetric* in a game (N, v) if their marginal contributions to every coalition coincide, i.e.

if for every $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$. A game, (N, v), is called *monotone* if for every $S, T \in 2^N$ with $S \subseteq T$, $v(S) \leq v(T)$.

Definition 2.1 A simple game is a monotone game such that the worth of every coalition is either 0 or 1 and the worth of the grand coalition is 1. Formally, (N, v) is a simple game if and only if:

- (N, v) is monotone,
- for every $S \in 2^N$, $v(S) \in \{0, 1\}$, and
- $v(N) = 1$.

The class of all simple games is denoted by $\mathcal{SG}$.

In a simple game $(N, v) \in \mathcal{SG}$, a coalition $S \subseteq N$ is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. $W(v)$ denotes the set of winning coalitions of the simple game (N, v) and, given $i \in N$, $W_i(v)$ denotes the subset of $W(v)$ formed by the coalitions containing player i, i.e. $W_i(v) = \{S \in W(v) : i \in S\}$. Given a simple game $(N, v) \in \mathcal{SG}$, a swing for a player $i \in N$ is a coalition $S \in 2^N$ such that $i \in S$, S is a winning coalition and $S \setminus i$ is a losing coalition. The set of all swings for player $i \in N$ is denoted by $\eta_i(v)$. Any simple game $(N, v) \in \mathcal{SG}$ may be described by its set of winning coalitions $W(v)$. Given a player set N and an arbitrary family of coalitions $W \subseteq 2^N$, the pair (N, W) determines a simple game if:

- $\emptyset \notin W$,
- $N \in W$ and
- for every $S \subseteq T \subseteq N$, if $S \in W$, then $T \in W$.

A winning coalition $S \in W(v)$ is a *minimal winning* coalition if every proper subset of S is a losing coalition; that is, S is a minimal winning coalition in (N, v) if $v(S) = 1$ and $v(T) = 0$ for any $T \subsetneq S$. $W^m(v)$ denotes the set of minimal winning coalitions of the game (N, v) and $W_i^m(v)$ the subset of $W^m(v)$ formed by coalitions containing player i, i.e. $W_i^m(v) = \{S \in W^m(v) : i \in S\}$. Similar to the case of winning coalitions, a simple game may also be defined by its set of minimal winning coalitions $W^m(v)$. Given a player set N and an arbitrary family of coalitions $W^m \subseteq 2^N$, the pair (N, W^m) determines a simple game if:

- $\emptyset \notin W^m$,
- $W^m \neq \emptyset$ and
- for every $S, T \in W^m$, $S \not\subset T$ and $T \not\subset S$.

It is easy to obtain the set of minimal winning coalitions from the set of winning coalitions and vice versa, i.e.

$$W^m(v) = \{S \in W(v) : \forall T \not\subset S, \quad T \notin W(v)\},$$
$$W(v) = \{S \in 2^N : \quad \exists T \subseteq S, \quad T \in W^m(v)\}.$$

By a *power index* we mean a map f that assigns a vector $\mathsf{f}(N, v) \in \mathbb{R}^{|N|}$ to every simple game $(N, v) \in \mathcal{SG}$. In the definitions below, two of the most popular power indices are presented.

Definition 2.2 The *Shapley–Shubik power index* [3], SS, is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{SS}_i(N, v) = \sum_{S \in \eta_i(v)} \frac{s!(n-s-1)!}{n!},$$

where $n = |N|$ and $s = |S|$.

Definition 2.3 The *Penrose–Banzhaf–Coleman power index* [4,11,12], PBC, is the power index defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by

$$\mathsf{PBC}_i(N, v) = \frac{|\eta_i(v)|}{2^{n-1}}.$$

In order to present characterizations of SS and PBC formally, some properties need to be presented.

EFF A power index f satisfies *efficiency* if for every $(N, v) \in \mathcal{SG}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v) = 1.$$

NPP A power index f satisfies the *null player property* if for every $(N, v) \in \mathcal{SG}$ and each null player $i \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = 0.$$

SYM A power index f satisfies *symmetry* if for every $(N, v) \in \mathcal{SG}$ and each pair of symmetric players $i, j \in N$ in (N, v),

$$\mathsf{f}_i(N, v) = \mathsf{f}_j(N, v).$$

TRP A power index f satisfies the *transfer property* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$,

$$\mathsf{f}(N, v) + \mathsf{f}(N, w) = \mathsf{f}(N, v \vee w) + \mathsf{f}(N, v \wedge w),$$

where $(N, v \vee w), (N, v \wedge w) \in \mathcal{SG}$ are defined for all $S \subseteq N$ by $(v \vee w)(S) = \max\{v(S), w(S)\}$ and $(v \wedge w)(S) = \min\{v(S), w(S)\}$.

TPP A power index f satisfies the *total power property* if for every $(N, v) \in \mathcal{SG}$,

$$\sum_{i \in N} \mathsf{f}_i(N, v) = \frac{\sum_{i \in N} |\eta_i(v)|}{2^{n-1}}.$$

Next, in line with the characterizations of the Shapley and Banzhaf values by [13], parallel characterizations of SS and PBC are presented.

THEOREM 2.4 [14] *The Shapley–Shubik power index,* SS, *is the unique power index satisfying* EFF, SYM, NPP *and* TRP.

THEOREM 2.5 [15] *The Penrose–Banzhaf–Coleman power index,* PBC, *is the unique power index satisfying* TPP, SYM, NPP *and* TRP.

The main difference between the Shapley value and the Banzhaf value is that the former is efficient while the later satisfies the total power property. The characterizations above show that this difference is transferred when simple games are considered. Hence, the main difference between SS and PBC is that the former is efficient while the latter satisfies the total power property.

3. Power indices based on minimal winning coalitions

The Deegan–Packel power index [5] is based on the idea that when it comes to measure the power of an agent, it should only be considered his participation in minimal winning coalitions. Moreover, it assumes the following three facts:

- Only minimal winning coalitions will emerge victorious.
- Each minimal winning coalition has an equal probability of forming.
- Players in minimal winning coalitions divide the spoils equally.

The conditions above seem reasonable in many situations. The first condition is a consequence of having rational players in the sense that they seek for maximizing their power and hence, they will only participate in minimal winning coalitions. In other words, if a winning coalition is not a minimal winning coalition, it means that there are players whose participation in the coalition is not needed. Hence, the remaining players will prefer to form the minimal coalition contained on the winning coalition since there will be less people to share the spoils with. The second condition states that all minimal winning coalitions are equally likely, which is very reasonable once the first condition is accepted. The last condition is a solidarity or equal treatment property. The requisites above lead to the following definition.

Definition 3.1 Given a simple game $(N, v) \in \mathcal{SG}$, the *Deegan–Packel power index* [5], DP, is a vector in $\mathbb{R}^{|N|}$ where each coordinate $(i \in N)$ is defined as follows:

$$\mathsf{DP}_i(N, v) = \frac{1}{|W^m(v)|} \sum_{S \in W_i^m(v)} \frac{1}{|S|}.$$

The DP power index is introduced in [5] together with a characterization by means of four properties. The characterization of SS presented before shares three of them, namely, EFF, SYM and NPP. Indeed, DP coincides with SS in the class of unanimity games. However, the DP power index does not satisfy TRP. Instead, it satisfies the so-called DP-mergeability property that is introduced next.

Two simple games (N, v) and (N, w) are *mergeable* if for every pair of minimal winning coalitions $S \in W^m(v)$ and $T \in W^m(w)$, it holds that $S \not\subset T$ and $T \not\subset S$. If two games (N, v) and (N, w) are mergeable, the minimal winning coalitions in the maximum game $(N, v \vee w)$ are precisely the union of the minimal winning coalitions in the two original games (N, v) and (N, w). Hence, the mergeability condition guarantees that $|W^m(v \vee w)| = |W^m(v)| + |W^m(w)|$. Recall the definition of a *merged or maximum game*, $(N, v \vee w)$, given for every $S \subseteq N$ by, $(v \vee w)(S) = \max\{v(S), w(S)\}$.

DP-MER A power index f satisfies *DP-mergeability* if for every pair of mergeable simple games $(N, v), (N, w) \in \mathcal{SG}$,

$$\mathsf{f}(N, v \vee w) = \frac{|W^m(v)| \cdot \mathsf{f}(N, v) + |W^m(w)| \cdot \mathsf{f}(N, w)}{|W^m(v \vee w)|}.$$

The property above states that the power in a merged game is a weighted mean of the power in the two component games. The weights come from the number of minimal winning coalitions in each component game, divided by the number of minimal winning coalitions in the merged game. Hence, it coincides with TRP in the sense that it assesses the power in a merged game in terms of the power in the two component games. At this point, the properties needed to present the first characterization of the Deegan–Packel index have been introduced.

THEOREM 3.2 [5] *The Deegan–Packel index,* DP, *is the unique power index satisfying* EFF, SYM, NPP *and* DP-MER.

More recently, [9] proposed a different characterization of the Deegan–Packel index. This characterization is based on the so-called DP-minimal monotonicity property which is inspired by the strong monotonicity property introduced in [2] to characterize the Shapley value. The property is formally introduced next.

DP-MM A power index f satisfies *DP-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \cdot |W^m(v)| \leq \mathsf{f}_i(N, w) \cdot |W^m(w)|.$$

The DP-MM property states that if the minimal winning coalitions that contain a player $i \in N$ are minimal winning coalitions in another game, then the power of player i in the former game times its number of minimal winning coalitions is never bigger than the power of the player in the latter game times its number of minimal winning coalitions. Hence, DP-MM describes the way in which the power of an agent changes when his position in the simple game is improved. The characterization of DP power index proposed in [9] replaces the DP-MER property by the DP-MM property.

THEOREM 3.3 [9] *The Deegan–Packel index,* DP, *is the unique power index satisfying* EFF, SYM, NPP *and* DP-MM.

In the scientific literature concerning power indices, one can find another power index that takes only minimal winning coalitions into account. The so-called Public Good index proposed in [6] considers that each player's power is proportional to the amount of minimal winning coalitions in which he participates.

Definition 3.4 Given a simple game $(N, v) \in \mathcal{SG}$, the *Public Good index* [6], PG, is a vector in $\mathbb{R}^{|N|}$ where each coordinate $(i \in N)$ is defined as follows:

$$\mathsf{PG}_i(N, v) = \frac{|W_i^m(v)|}{\sum_{j \in N} |W_j^m(v)|}.$$

The first characterization of this power index by means of a set of properties is proposed in [16]. The characterization follows the spirit of the characterization of the DP index presented in Theorem 3.2. Indeed, the property of DP-MER is replaced by the PG-MER which is formally introduced below.

PG-MER A power index f satisfies *PG-mergeability* if for every pair of mergeable simple games $(N, v), (N, w) \in \mathcal{SG}$ and every $i \in N$,

$$\mathsf{f}_i(N, v \vee w) = \frac{\mathsf{f}_i(N, v) \cdot \sum_{j \in N} |W_j^m(v)| + \mathsf{f}_i(N, w) \cdot \sum_{j \in N} |W_j^m(w)|}{\sum_{j \in N} |W_j^m(v \vee w)|}.$$

Hence, PG-MER describes the power in the merged game as a weighted mean of the powers in the two component games as the DP-MER does. However, the weights used differ from the ones used in DP-MER. Next, the counterpart of Theorem 3.2 for the PG index is presented.

THEOREM 3.5 [16] *The Public Good index,* PG*, is the unique power index satisfying* EFF, SYM, NPP *and* PG-MER.

More recently, [10] proposed a different characterization of the Public Good index. This characterization is parallel to the one for the DP index presented in Theorem 3.3. It is based on the so-called PG-minimal monotonicity property which is similar to the DP-minimal monotonicity property stated above. The property is formally introduced next.

PG-MM A power index f satisfies *PG-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \cdot \sum_{j \in N} |W_j^m(v)| \leq \mathsf{f}_i(N, w) \cdot \sum_{j \in N} |W_j^m(w)|.$$

The PG-MM property keeps a close relation with the PG-MM property. Both properties describe the relation between the power of an agent in two different simple games when the minimal winning coalitions that contain the player in one game are minimal winning coalitions in the other game. The difference lies on the scalars that multiply the power in each of the simple games. Hence, using PG-MM property, a counterpart of Theorem 3.3 is obtained for the PG index.

THEOREM 3.6 [10] *The Public Good index,* PG*, is the unique power index satisfying* EFF, SYM, NPP *and* PG-MM.

The four characterization results presented in this section are summarized in Table 1.

4. Two new power indices based on null player free winning coalitions

In Section 2, the SS and PBC power indices are introduced. These indices are based on the swings of each player, that is, on the winning coalitions containing the player that become loosing when the player leaves them. In Section 3, power indices based on minimal winning coalitions are introduced, namely DP and PG. In this section, two new power indices are introduced following [1].

A winning coalition $S \in W(v)$ is said to be a *null player free winning coalition* if no null player is contained on it, that is, if for every $i \in S$ there is $T \in W_i(v)$ such that $T \setminus i \notin W(v)$. The set of null player free winning coalitions is denoted by $W^{np}(v)$. As before, for every

Table 1. Parallel characterizations of DP and PG.

	DP	PG	
[5]	DP-MER EFF SYM NPP	PG-MER EFF SYM NPP	[16]
[9]	DP-MM EFF SYM NPP	PG-MM EFF SYM NPP	[10]

player $i \in N$, $W_i^{np}(v)$ denotes the set of null player free winning coalitions containing player i, i.e. $W_i^{np}(v) = \{S \in W^{np}(v) : i \in S\}$. Note that for every $(N, v) \in \mathcal{SG}$, the following relation holds,

$$W^m(v) \subseteq W^{np}(v) \subseteq W(v).$$

Thus, the set of null player free winning coalitions can be seen either as a refinement of the set of winning coalitions or as an extension of the set of minimal winning coalitions.

Note that a simple game is determined by its set of null player free winning coalitions, $W^{np}(v)$. The claim before holds since the set of winning coalitions can be easily obtained from $W^{np}(v)$, i.e.

$$W(v) = \{T \in 2^N : \text{there is } S \in W^{np}(v) \text{ such that } S \subseteq T\}.$$

It is also easy to obtain the set of minimal winning coalitions given the set of null player free winning coalitions and vice versa, as follows,

$$W^m(v) = \{T \in W^{np}(v) : \text{for every } S \subsetneq T, \quad S \notin W^{np}(v)\}. \tag{1}$$

$$W^{np}(v) = \{S \subseteq \bigcup_{U \in W^m(v)} U : \text{there is } T \in W^m(v) \text{ such that } T \subseteq S\} \tag{2}$$

In [1] two new power indices based on null player free winning coalitions are introduced. In the paper, the new power indices are denoted by f and g. However, the notation is slightly modified here for the sake of clarity. The new power indices consider that only null player free winning coalitions should be taken into account when it comes to measuring the power. In other words, these power indices are based on the information contained on the set W^{np}. In this way, null players, which by definition do not participate in coalitions of W^{np}, are assigned no power. The formal definitions are introduced next.

Definition 4.1 The f^{np} power index is defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by,

$$\mathsf{f}_i^{np}(N, v) = \frac{1}{|W^{np}(v)|} \sum_{S \in W_i^{np}(v)} \frac{1}{|S|}.$$

Definition 4.2 The g^{np} power index is defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by,

$$\mathsf{g}_i^{np}(N, v) = \frac{|W_i^{np}(v)|}{\sum_{j\in N} |W_j^{np}(v)|}.$$

The idea behind the power indices defined above is in line with the definitions of the Deegan–Packel and the Public Good indices (see Definitions 3.1 and 3.4). The only difference is that f^{np} and g^{np} consider all winning coalitions that do not contain null players instead of only considering minimal winning coalitions. A direct consequence of this fact is that non-null players which do not participate in many minimal winning coalitions are allocated more power.

Consequently, f^{np} considers that all null player free winning coalitions are equally likely and that the players in a null player free winning coalition divide the spoils equally. g^{np} assumes that the power of each player is proportional to the number of null player free winning coalitions in which he participates. The Shift power index introduced in [8] is similar to the Public Good index and the g^{np} power index; however, it is based on a set of coalitions which is contained on the set of minimal winning coalitions.

A set of few independent properties is a convenient tool to describe a power index and eases the comparison among different power indices. In order to characterize f^{np} and g^{np}, the following monotonicity properties are introduced.

f^{np}-MM A power index f satisfies f^{np}*-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \cdot |W^{np}(v)| \leq \mathsf{f}_i(N, w) \cdot |W^{np}(w)|.$$

g^{np}-MM A power index f, satisfies g^{np}*-minimal monotonicity* if for every pair of simple games $(N, v), (N, w) \in \mathcal{SG}$ and every player $i \in N$ such that $W_i^m(v) \subseteq W_i^m(w)$,

$$\mathsf{f}_i(N, v) \sum_{j\in N} |W_j^{np}(v)| \leq \mathsf{f}_i(N, w) \sum_{j\in N} |W_j^{np}(w)|.$$

The f^{np}-MM and g^{np}-MM properties are based on the strong monotonicity property used in [2] to characterize the Shapley value. Indeed, both describe the behaviour of a value in two simple games, (N, v) and (N, w), in which there is a player $i \in N$ such that $W_i(v) \subseteq W_i(w)$, in other words, $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$ for every $S \subseteq N \setminus i$. The difference lies on the relation between the power of player i in both games. The strong monotonicity property states that player i's power in (N, w) is at least as big as his power in (N, v). Instead, f^{np}-MM and g^{np}-MM properties state that the relation holds after multiplying the payoffs by the denominator of the definitions of f^{np} and g^{np} respectively.

The following results show that f^{np} and g^{np} are characterized with a close set of properties of the ones used in Theorems 3.3 and 3.6 to characterize DP and PG, respectively.

THEOREM 4.3 *The power index* f^{np} *is the unique power index that satisfies* EFF, NPP, SYM *and* f^{np}-MM.

Proof (1) *Existence*. From Definition 4.1, it straightforward to check that f^{np} satisfies EFF, NPP and SYM. For f^{np}-MM property, note that by Equation (2), $W_i^m(v) \subseteq W_i^m(w)$

implies $W_i^{np}(v) \subseteq W_i^{np}(w)$. Then,

$$\begin{aligned} \mathsf{f}_i^{np}(N, w) &= \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(w)} \frac{1}{|S|} \\ &= \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} + \frac{1}{|W^{np}(w)|} \sum_{S \in W_i^{np}(w) \setminus W_i^{np}(v)} \frac{1}{|S|}, \end{aligned}$$

and hence,

$$\begin{aligned} &\mathsf{f}_i^{np}(N, w) \cdot |W_i^{np}(w)| \\ &\quad = \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} + \sum_{S \in W_i^{np}(w) \setminus W_i^{np}(v)} \frac{1}{|S|} \geq \sum_{S \in W_i^{np}(v)} \frac{1}{|S|} \\ &\quad = \mathsf{f}_i^{np}(N, v) \cdot |W_i^{np}(v)|. \end{aligned}$$

(2) *Uniqueness*. The uniqueness is proved by induction on the number of minimal winning coalitions. If $|W^m(v)| = 1$, then $v = u_S$ where $W^m(v) = \{S\}$. If a power index, f satisfies EFF, NPP and SYM, we have,

$$\mathsf{f}_i(N, v) = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}.$$

Hence, the uniqueness holds when $|W^m(v)| = 1$. Next, assume that a power index satisfying the properties is unique for every $(N, v) \in \mathcal{SG}$ with less than $m > 1$ minimal winning coalitions, i.e., f is unique for every $(N, v) \in \mathcal{SG}$ such that $|W^m(v)| < m$. Let $(N, v) \in \mathcal{SG}$ with $W^m(v) = \{S_1, \ldots, S_m\}$. Take $T = \cap_{k=1}^m S_k$. Then, for each $i \notin T$ let us define $(N, w) \in \mathcal{SG}$ by $W^m(w) = W_i^m(v)$. Then, since $W_i^m(v) = W_i^m(w)$, applying f^{np}-MM twice,

$$\mathsf{f}_i(N, v)|W^m(v)| = \mathsf{f}_i(N, w)|W^m(w)|.$$

Finally, note that $|W^m(w)| < m$ and hence, by induction, the right hand side of the equality above is unique. It remains to prove the uniqueness for $i \in T$. By SYM, there is a constant $c \in \mathbb{R}$ such that for every $i \in T$, $\mathsf{f}_i(N, v) = c$. Moreover, by EFF and the uniqueness for any $i \notin T$, c is unique which concludes the proof. □

THEOREM 4.4 *The power index g^{np} is the unique power index that satisfies* EFF, NPP, SYM, *and* g^{np}-MM.

Proof The proof follows immediately from a reasoning similar to the one used in the Proof of Theorem 4.3. □

Hence, the two Theorems above show that the differences between f^{np} and g^{np} are restricted to a monotonicity property. Moreover, the only difference among SS, DP, PG, f^{np} and g^{np} is the type of monotonicity satisfied by each power index. Finally, the parallel characterizations of f^{np} and g^{np} are depicted in Table 2.

5. Conclusion

In a simple game, a member is considered critical for a winning coalition when his elimination from the coalition turns it into a losing one. In a minimal winning coalition, every

Table 2. Parallel characterizations of f^{np} and g^{np}.

f^{np}	g^{np}
f^{np}-MM	g^{np}-MM
EFF	EFF
SYM	SYM
NPP	NPP

member of the coalition is critical. The Deegan–Packel and Public Good indices are based on minimal winning coalitions. A player is null when it is not critical for any winning coalition. However, in most of the cases, a null player participates in many winning coalitions. For this reason, the new indices, f^{np} and g^{np}, are based on winning coalitions that do not contain null players. The characterizations provided in this paper highlight the fact that the new power indices share most of their defining features with other well-known indices.

The project initiated with this paper is not closed and we have related future work in mind. The two main lines of this future work are to extend these indices to more complex models and to propose tools to compute these indices.

One of the models developed for representing decision-making bodies more adequately is the one of simple games with a coalition structure. In [17,19] extensions of the Deegan–Packel and Public Good indices are proposed and characterized for simple games with a coalition structure. The indices characterized in this paper can be extended to the model of simple games with a coalition structure. Moreover, we also want to consider more complex models like games with graph-restricted communication or games with levels structure of cooperation.

Although the mathematical expression of the indices characterized in this paper is simple, one of the difficulties is their computation for games with a big number of players. Two tools have been used to compute power indices in large games: the multilinear extensions [20,21] and [10] and the generating functions [22]. One of the ideas that we want to develop in the near future is to find methods which allow to compute the proposed indices by means of these two tools.

Acknowledgements

A. A. Pinto acknowledges the financial support of LIAAD INESC TEC through program PEst, USP-UP project, Faculty of Sciences, University of Porto, Calouste Gulbenkian Foundation, FEDER, POCI 2010 and COMPETE Programmes, PTDC/MAT/121107/2010 and Fundação para a Ciência e a Tecnologia (FCT). J. M. Alonso-Meijide and M. Álvarez-Mozos acknowledge the financial support of the Spanish Ministry of Economy and Competitiveness through Projects ECO2008-03484-C02-02/ECO and MTM2011-27731-C03-03 and of Xunta de Galicia through Project INCITE09-207-064-PR. The authors would also like to thank the comments and suggestions of two anonymous referees.

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