Comput Stat (2012) 27:69–82
DOI 10.1007/s00180-011-0238-4

ORIGINAL PAPER

Goodness-of-fit for a concentrated von Mises-Fisher distribution

Adelaide Maria Sousa Figueiredo

Received: 29 May 2008 / Accepted: 1 February 2011 / Published online: 16 February 2011

Abstract The von Mises-Fisher distribution is widely used for modelling directional data. In this paper we propose goodness-of-fit methods for a concentrated von Mises-Fisher distribution and we analyse by simulation some questions concerning the application of these tests. We analyse the empirical power of the Kolmogorov-Smirnov test for several dimensions of the sphere, supposing as alternative hypothesis a mixture of two von Mises-Fisher distributions with known parameters. We also compare the empirical power of the Kolmogorov-Smirnov test with the Rao's score test for data on the sphere, supposing as alternative hypothesis, a mixture of two Fisher distributions with unknown parameters replaced by their maximum likelihood estimates or a 5-parameter Fisher-Bingham distribution. Finally, we give an example with real spherical data.

Keywords Directional data · Fisher-Bingham distribution · Goodness-of-fit test · Von Mises-Fisher distribution

Mathematics Subject Classification (2000) 62H11 · 62G10

1 Introduction

The von Mises-Fisher distribution is frequently used for modelling directional data (see for instance, Watson 1983, Fisher et al. 1987, Fisher 1993 and Mardia and Jupp 2000). This distribution is called Fisher distribution for data on the sphere and is called von Mises distribution for data on the circle. Some goodness-of-fit tests have

A. M. S. Figueiredo (✉)
Matemática e Sistemas de Informação,
Faculdade de Economia/LIAAD—INESC Porto L.A., Universidade do Porto,
Rua Dr. Roberto Frias, 4200-464 Porto, Portugal
e-mail: adelaide@fep.up.pt

been proposed in the literature for this distribution in the case of circular data and spherical data. For the particular case of data defined on the circle, Lockhart and Stephens (1985) gave goodness-of-fit tests for the von Mises distribution and Lawson (1988) considered the fit of the von Mises distribution using GLIM. For the particular case of data defined on the sphere, Lewis and Fisher (1982) presented graphical methods for investigating the fit of a Fisher distribution to spherical data; Fisher and Best (1984) considered goodness-of-fit tests based on the empirical distribution function to investigate the adequacy of fit of Fisher's distribution on the sphere and Rivest (1986) proposed a goodness-of fit test for the Fisher distribution in small concentrated samples. Mardia et al. (1984) derived a likelihood ratio test for the adequacy of a von Mises-Fisher distribution against the alternative of a Fisher-Bingham distribution and Boulerice and Ducharme (1997) proposed for directional and axial data, general purposes smooth tests of goodness-of-fit for rotationally symmetric distributions, including the von Mises-Fisher distribution against general families of embedding alternatives constructed from complete orthonormal bases of functions. In Sect. 2 we recall some distributions used for directional data, namely the von Mises-Fisher distribution and the 5-parameter Fisher-Bingham distribution. In Sect. 3 we propose some goodness-of-fit methods for a concentrated von Mises-Fisher distribution based on an asymptotic chi-square distribution and we also recall a goodness-of-fit test based on the Rao's score statistic. In Sect. 4 we carry out a simulation study for analysing some questions concerning the application of the methods suggested in this paper: the adequacy of using the asymptotic chi-square distribution in the tests and the adequacy of using the tabulated critical values of the Kolmogorov-Smirnov statistic when the parameters of the von Mises-Fisher distribution are unknown and replaced by their maximum likelihood estimates. We also determine by simulation the critical values of the Rao's score statistic in some cases and we compare them with the asymptotic values. In Sect. 5 we analyse the empirical power of the Kolmogorov-Smirnov test for several dimensions of the sphere, supposing as an alternative hypothesis a mixture of two von Mises-Fisher distributions. We also compare the empirical power of this test with the Rao's score test for data on the sphere supposing as alternative hypothesis a mixture of two Fisher distributions with unknown parameters replaced by their maximum likelihood estimates or a Kent distribution. In Sect. 6 we consider spherical data given in the literature for illustrating the previous methods and finally, in Sect. 7, we conclude the paper.

2 Some distributions for directional data

The von Mises-Fisher distribution on the unit sphere in $\mathbb{R}^p$, $S_{p-1}=\left\{\mathbf{x} \in \mathbb{R}^p : \mathbf{x}'\mathbf{x}=1\right\}$ is usually denoted by $M_p\left(\boldsymbol{\mu}, \kappa\right)$ and it has probability density function given by

$$f\left(\mathbf{x}|\boldsymbol{\mu}, \kappa\right) = c_p\left(\kappa\right) \exp\left(\kappa \mathbf{x}'\boldsymbol{\mu}\right) \quad \mathbf{x} \in S_{p-1}, \quad \boldsymbol{\mu} \in S_{p-1},\ \kappa \geq 0\,, \tag{2.1}$$

where the normalising constant $c_p\left(\kappa\right)$ is defined by

$$c_p(\kappa) = \left(\frac{\kappa}{2}\right)^{\frac{p}{2}-1} \frac{1}{\Gamma\left(\frac{p}{2}\right) I_{\frac{p}{2}-1}(\kappa)}$$

and I_ν denotes the modified Bessel function of the first kind and order ν. For more details about this function, see for instance, Mardia and Jupp (2000), p. 168.

The parameter $\boldsymbol{\mu}$ is the mean direction and κ is the concentration parameter around $\boldsymbol{\mu}$. The parameter $\boldsymbol{\mu}$ is also the mode and $-\boldsymbol{\mu}$ is the antimode (provided that $\kappa > 0$). This distribution is rotationally symmetric about $\boldsymbol{\mu}$ (Mardia and Jupp 2000, p. 179). If $\mathbf{x}$ comes from $M_p(\boldsymbol{\mu}, \kappa)$ distribution and U is an orthogonal matrix, then $U\mathbf{x}$ comes from $M_p(U\boldsymbol{\mu}, \kappa)$ distribution. If $\mathbf{x}$ comes from $M_p(\boldsymbol{\mu}, \kappa)$ distribution then for large κ (Mardia and Jupp 2000, p. 172):

$$2\kappa\left(1 - \mathbf{x}'\boldsymbol{\mu}\right) \dot{\sim} \chi^2_{p-1}. \tag{2.2}$$

Let $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ be a random sample of size n from the von Mises-Fisher distribution $M_p(\boldsymbol{\mu}, \kappa)$. Let $\overline{R}_n$ be the resultant length mean of the sample defined by $\overline{R}_n = \left(\overline{\mathbf{x}}'\overline{\mathbf{x}}\right)^{1/2}$, where $\overline{\mathbf{x}}$ is the sample vector mean of $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ defined by $\overline{\mathbf{x}} = \sum_{i=1}^n \mathbf{x}_i / n$. The maximum likelihood estimator of $\boldsymbol{\mu}$ is the sample mean direction, that is $\widehat{\boldsymbol{\mu}} = \overline{\mathbf{x}}_0 = \overline{\mathbf{x}} / \overline{R}_n$ and the maximum likelihood estimator of κ is the solution of the equation

$$A_p(\kappa) = \overline{R}_n, \tag{2.3}$$

where the function $A_p(\kappa)$ is defined by $A_p(\kappa) = -c_p'(\kappa) / c_p(\kappa) = I_{\frac{p}{2}}(\kappa) / I_{\frac{p}{2}-1}(\kappa)$. (See Mardia and Jupp 2000, p. 198).
For $p = 3$, the von Mises-Fisher distribution is called Fisher distribution and denoted by $M_3(\boldsymbol{\mu}, \kappa)$ or by $F(\boldsymbol{\mu}, \kappa)$. For $p = 2$ the von Mises-Fisher distribution is called von Mises distribution and denoted by $M_2(\boldsymbol{\mu}, \kappa)$ or by $M(\boldsymbol{\mu}, \kappa)$.
The Fisher distribution is a particular case of the 5-parameter Fisher-Bingham distribution or Kent distribution proposed by Kent (1982). This distribution, denoted by $FB_5(\kappa, \beta, \Gamma)$ is defined by the density

$$f(\mathbf{x}) = c(k, \beta)^{-1} \exp\left\{k\boldsymbol{\gamma}'_{(1)}\mathbf{x} + \beta\left[\left(\boldsymbol{\gamma}'_{(2)}\mathbf{x}\right)^2 - \left(\boldsymbol{\gamma}'_{(3)}\mathbf{x}\right)^2\right]\right\} \quad \mathbf{x} \in S_2. \tag{2.4}$$

The parameters are the concentration $k \geq 0$, the ovalness $\beta \geq 0$ and a (3×3) ortogonal matrix $\Gamma = \left(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(2)}, \boldsymbol{\gamma}_{(3)}\right)$, where $\boldsymbol{\gamma}_{(1)}$ is the mean direction or pole, $\boldsymbol{\gamma}_{(2)}$ is the major axis and $\boldsymbol{\gamma}_{(3)}$ is the minor axis.

If $\beta = 0$, then (2.3) reduces to a Fisher density.

The FB_5 distribution is a spherical analogue of the bivariate normal distribution. More details about the FB_5 distribution can be found in Kent (1982).

3 Goodness-of-fit methods

We wish to test the null hypothesis H_0: The sample $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ comes from a von Mises-Fisher distribution $M_p(\boldsymbol{\mu}, \kappa)$, with large κ.
Based on the result (2.2), we suggest to reduce the goodness-of-fit of a von Mises-Fisher distribution to the goodness-of-fit of a chi-square distribution. For each $\mathbf{x}_i$, $i = 1, \ldots, n$ of the sample we calculate the value $y_i = 2\kappa\left(1 - \mathbf{x}_i'\boldsymbol{\mu}\right)$ and we test the null hypothesis H_0': The sample $(y_1, \ldots, y_n)$ comes from a $\chi^2_{(p-1)}$ population. Then, for a large concentration parameter κ, to test H_0' we may use a chi-square Q–Q plot and the usual chi-square and Kolmogorov-Smirnov (K.-S.) goodness-of-fit tests.
If the parameters $\boldsymbol{\mu}$ and κ are unknown, they are replaced by their maximum likelihood estimates.

Mardia et al. (1984) derive a goodness-of-fit test for the von Mises-Fisher distribution. These authors consider the likelihood ratio test for the null hypothesis of a Fisher distribution against the alternative of a Fisher-Bingham distribution and use the Rao's statistic, which is asymptotically equivalent to the likelihood ratio statistic, to test this hypothesis. This test takes a simpler form for $p = 3$ with an alternative of a Kent distribution (see Kent 1982; Mardia and Jupp 2000, pp. 272–273) and as we'll use it in this paper, next we'll describe it briefly.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be observations on the sphere S_2. Let $H\left(\overline{\mathbf{x}}_0, (0, 0, 1)'\right)$ be the rotation matrix, which transforms the sample mean direction $\overline{\mathbf{x}}_0$ into the pole $(0, 0, 1)'$. Denote by $\mathbf{y}_i$ the 2-vector defined by the last two components of $H\left(\overline{\mathbf{x}}_0, (0, 0, 1)'\right)\mathbf{x}_i$. Let $\widehat{l_1}$ and $\widehat{l_2}$ be the eigenvalues of

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}_i\mathbf{y}_i'.$$

Then the score statistic takes the form

$$\widehat{T} = n\frac{\widehat{\kappa}^3}{4\left(\widehat{\kappa} - 3\overline{R}\right)}\left(\widehat{l_1} - \widehat{l_2}\right)^2, \tag{3.1}$$

where $\widehat{\kappa}$ is the maximum likelihood estimate of κ under the Fisher distribution.

The null hypothesis of a Fisher distribution is rejected for large values of $\widehat{T}$. Under the null hypothesis, the large-sample asymptotic distribution of $\widehat{T}$ is

$$\widehat{T} \dot{\sim} \chi^2_{(2)}.$$

4 Simulation study

In this study we have analysed some questions concerning the application of the tests of the previous section. The first question is to determine, for several dimensions of the sphere p, the values of κ for which the approximation (2.2) is valid. For investigating this, we compare, for each dimension of the sphere p, the empirical distribution of

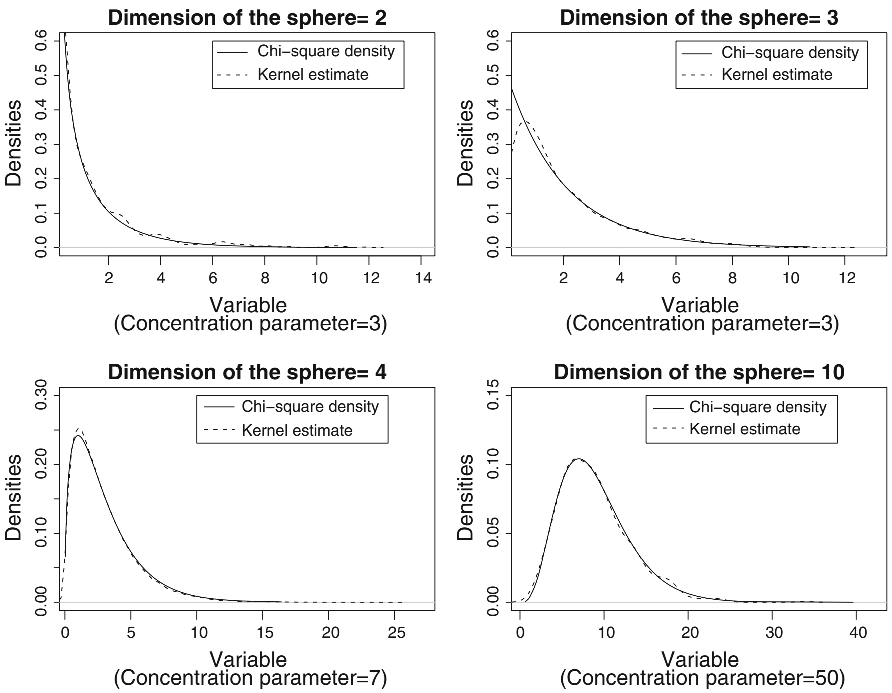

Fig. 1 Kernel estimate and $\chi^2_{(p-1)}$ distribution for $p = 2, 3, 4, 10$

$Y = 2\kappa\left(1 - \mathbf{x}'\boldsymbol{\mu}\right)$, obtained from 1,000 generated values of Y with the χ^2_{p-1} distribution for several values of κ. We note that for each generated value of Y a vector $\mathbf{x}$ was simulated from the von Mises-Fisher distribution using the method proposed in Wood (1994) and we also note that we have supposed without loss of generality that the mean direction in the von Mises-Fisher distribution is equal to $\boldsymbol{\mu} = \mathbf{e}_p = (0, 0, \ldots, 0, 1)'$. Should we use another mean direction instead of $\mathbf{e}_p$, we would obtain the same results. More precisely, we have compared for several values of κ, the kernel estimate of the density function with the χ^2_{p-1} density function for $p = 2, 3, 4, 10$, respectively. We have observed that for each p, the χ^2_{p-1} distribution seems to be very adequate even for a small value of κ, that is, for $\kappa \geq 3$, when $p = 2$ and $p = 3$, for $\kappa \geq 7$ when $p = 4$ and for $\kappa \geq 50$ when $p = 10$. See Fig. 1 We note that the densities in this figure were obtained in R and we have used a gaussian smoothing for the kernel estimate.

The second question is to analyse by simulation the critical values of Kolmogorov-Smirnov statistic under the chi-square distribution we are testing. When the parameters of the distribution we are testing are known, the critical values of the Kolmogorov-Smirnov statistic are tabulated. In our case, we are testing a chi-square distribution with known parameter and if the parameters of the von Mises-Fisher distribution are also known, we may use the critical values, which are tabulated. But if the parameters of the von Mises-Fisher distribution are unknown, the distribution of the Kolmogorov-Smirnov statistic does not depend on the parameters of the underlying von Mises-Fisher distribution, but depends on the fact that the y_i are approximately chi-square, since the parameters had been estimated. Then, it is important to verify the adequacy of using the tabulated critical values in this case.

We have generated 100,000 samples of size n from the von Mises-Fisher distribution for $p = 2, 3, 4, 10$ and for various values of κ for both cases: known parameters of the von Mises-Fisher distribution and unknown parameters of this distribution replaced by their maximum likelihood estimates. We have determined the upper 1, 5 and 10% percentiles of the Kolmogorov-Smirnov statistic, which are indicated for several values of n and κ in Tables 6 and 7 of Appendix for known parameters and unknown parameters, respectively.

Obviously, the critical values obtained by simulation in the case of known parameters are closer to the tabulated critical values than in the case of unknown parameters.

Finally, as we'll use in the next section the test based on the Rao's score statistic defined by (3.1), we have determined by simulation the critical values for this statistic and we have compared them with the asymptotic critical values. We have generated 100,000 samples of size $n = 20, 30, 50$ from the Fisher distribution for $\kappa = 5, 7, 10(10)50$. Then we have obtained 100,000 replicates of the statistic defined by (3.1) and we have determined the upper 1, 5 and 10% percentiles of the statistic, which are indicated in Table 8 of Appendix.

5 Empirical size and power of the tests

First, we have determined the empirical size and power of the Kolmogorov-Smirnov test for a null hypothesis of a von Mises-Fisher distribution against an alternative with equal proportions of a mixture of two von Mises-Fisher distributions, for various dimensions of the sphere. We consider the null hypothesis:

H_0 : The sample $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ comes from the von Mises-Fisher distribution $M_p(\mathbf{e}_1, \kappa)$,

where $\mathbf{e}_1 = (1, \ldots, 0, 0)'$ and κ is large.
Should we use another mean direction instead of $\mathbf{e}_1$, we would obtain the same results. Then this hypothesis is reduced to

H_0' : The values $y_i = 2\kappa\left(1 - \mathbf{x}_i'\mathbf{e}_1\right), i = 1, \ldots, n$ are $\chi^2_{(p-1)}$ distributed, for κ large. We consider as alternative hypothesis:

H_1 : The sample $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ comes from the mixture with equal proportions of two von Mises-Fisher distributions $M_p(\mathbf{e}_1, \kappa)$ and $M_p(\mathbf{e}_p, \kappa)$,

where $\mathbf{e}_p = (0, \ldots, 0, 1)'$ and the mean directions of the components of the mixture form an angle θ of 90°. We suppose that the parameters $\mathbf{e}_1$, $\mathbf{e}_p$ and κ are known.

Since the parameter of the chi-square distribution is known and the parameters of the von Mises-Fisher distributions are known in the Kolmogorov-Smirnov statistic, we use the tabulated critical values of this statistic to determine the size and the empirical power of the test. The tabulated critical values of the Kolmogorov-Smirnov statistic for a significance level of 5%, are equal to 0.409, 0.294, 0.242, 0.192, 0.136 for the dimensions of the sample equal to 10, 20, 30, 50, 100, respectively.

Then, we have determined the size of the test based on 100,000 replicates of the Kolmogorov-Smirnov statistic obtained generating 100,000 samples under the null

Table 1 Size of the K.-S. test supposing known parameters

p	$n\backslash\kappa$	3	5	7	10	20	30
2	10	0.056	0.053	0.052	0.052	0.051	0.051
	20	0.059	0.055	0.053	0.052	0.052	0.052
	30	0.063	0.054	0.052	0.051	0.051	0.051
3	10	0.050	0.051	0.051	0.050	0.051	0.051
	20	0.049	0.051	0.050	0.050	0.051	0.050
	30	0.050	0.050	0.050	0.049	0.050	0.051
	$n\backslash\kappa$	5	7	10	20	30	50
4	10	0.052	0.050	0.050	0.050	0.050	0.050
	20	0.057	0.052	0.052	0.050	0.049	0.050
	30	0.060	0.053	0.051	0.050	0.049	0.049
	$n\backslash\kappa$	50	60	70	80	90	100
10	10	0.056	0.053	0.052	0.052	0.051	0.051
	20	0.059	0.055	0.053	0.052	0.052	0.052
	30	0.063	0.054	0.052	0.051	0.051	0.051

hypothesis, supposing that the parameters of the von Mises-Fisher distribution are known. See Table 1. We note that for $p = 10$ we have considered larger values of κ than for $p = 2, 3, 4$ because we had concluded in Sect. 4 that the approximation (2.2) is valid only for $\kappa \geq 50$. We have observed that for the analysed cases, the empirical significance level is approximately equal to the nominal level of significance of 5%. Since we do not know the distribution of the test statistic under H_1, we have determined the empirical power of the test based on 100,000 replicates of the Kolmogorov-Smirnov statistic obtained under the alternative hypothesis, supposing that the parameters of the components of the mixture are known. See Table 2. We have concluded the following:

- For fixed n, the empirical power of the Kolmogorov-Smirnov test increases or remains constant when the concentration parameter κ increases. This is as expected because as κ increases, more concentrated are the von Mises-Fisher distribution around its mean direction and the components of the mixture around their respective mean directions. Consequently, it is easier to distinguish between the distributions given in H_0 and H_1, and then greater is the empirical power of the test.
- For fixed κ, the empirical power of the Kolmogorov-Smirnov test increases when the dimension of the sample n increases. This is also as expected, because as smaller is the sample size, easier is the fit of any distribution and smaller is the empirical power of the test.
- As expected for the same n and κ, the empirical power of the Kolmogorov-Smirnov test decreases when the dimension of the sphere p increases.

Second, we determine the size and the empirical power of the Kolmogorov-Smirnov test in the case of the sphere ($p = 3$) for the hypothesis of a Fisher distribution, supposing as alternatives a mixture with equal proportions of two Fisher distributions with

Table 2 Empirical power of the K.-S. test with an alternative of a mixture with known parameters

p	$n\backslash\kappa$	3	5	7	10	20	30
2	10	0.750	0.783	0.784	0.782	0.781	0.781
	20	0.960	0.978	0.984	0.987	0.987	0.988
	30	0.995	0.999	0.999	0.999	0.999	0.999
3	10	0.521	0.709	0.762	0.776	0.778	0.777
	20	0.801	0.946	0.974	0.95	0.988	0.987
	30	0.931	0.993	0.999	0.998	0.999	0.999
	$n\backslash\kappa$	5	7	10	20	30	50
4	10	0.622	0.727	0.763	0.772	0.774	0.775
	20	0.892	0.959	0.979	0.987	0.987	0.987
	30	0.975	0.995	0.998	0.999	0.999	0.999
	$n\backslash\kappa$	50	60	70	80	90	100
10	10	0.755	0.759	0.761	0.764	0.764	0.766
	20	0.985	0.985	0.986	0.986	0.986	0.986
	30	0.999	0.999	0.999	0.999	0.999	0.999

unknown parameters and a Kent distribution, and we have compared this test with the Rao's test based on the statistic defined by (3.1). The null hypothesis to test is:

H_0 : The sample $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ comes from the Fisher distribution $F(\mathbf{e}_1, \kappa)$, where $\mathbf{e}_1 = (1, 0, 0)'$ and κ is large. Then this hypothesis is reduced to
H_0' : The values $y_i = 2\kappa\left(1 - \mathbf{x}_i'\mathbf{e}_1\right), i = 1, \ldots, n$ are $\chi^2_{(2)}$ distributed, for κ large.

We obtain the empirical size of the two tests based on 100,000 replicates of the test statistics by generating 100,000 samples under H_0 and supposing that the parameters $\mathbf{e}_1$ and κ are unknown and replaced by their maximum likelihood estimates. Although we have used the estimates of the parameters in the Kolmogorov-Smirnov statistic, we have used the tabulated critical values for a level of significance of 5% of the Kolmogorov-Smirnov statistic because we have previously observed that the critical values obtained by simulation, indicated in Table 7 are relatively close to the tabulated ones. As the asymptotic distribution of the Rao's score statistic is known under the null hypothesis of a Fisher distribution: $\chi^2_{(2)}$, we have used the critical values for a 5% level of significance of this distribution. We note that if we had used the critical values obtained by simulation indicated in Tables 7 and 8 for the K.-S. and Rao's test, respectively, we would obtain similar results in each case for the size of the tests and also for the empirical power of the tests with the two alternative hypotheses. See Table 3. We observe that for the analysed cases, the size of the tests is approximately equal to the significance level of 5%. Next, we consider the alternative hypothesis:

H_1 : The sample comes from the mixture with equal proportions of two Fisher distributions $F(\mathbf{e}_1, \kappa)$ and $F(\mathbf{e}_p, \kappa)$,

where $\mathbf{e}_p = (0, 0, 1)'$. Since we do not know the distribution of the test statistics under the alternative distribution, we have estimated the power of the tests by generating 100,000 samples from this distribution. The empirical power obtained for

Table 3 Size of the tests supposing unknown parameters

n	κ	2	3	4	5	7	10	20
20	K-S	0.040	0.050	0.041	0.036	0.042	0.044	0.059
	Rao	0.053	0.064	0.061	0.069	0.074	0.077	0.083
30	K-S	0.035	0.050	0.045	0.033	0.041	0.041	0.048
	Rao	0.049	0.058	0.064	0.065	0.069	0.070	0.074
50	K-S	0.050	0.043	0.043	0.030	0.039	0.037	0.039
	Rao	0.047	0.055	0.060	0.060	0.064	0.064	0.065
100	K-S	0.032	0.039	0.044	0.035	0.040	0.035	0.031
	Rao	0.047	0.052	0.056	0.058	0.060	0.059	0.058

Table 4 Empirical power of the tests with an alternative of a mixture supposing unknown parameters

n	κ	2	3	4	5	7	10	20
20	K-S	0.689	0.863	0.935	0.960	0.980	0.988	0.992
	Rao	0.171	0.451	0.737	0.896	0.988	0.999	1
30	K-S	0.772	0.939	0.982	0.993	0.998	0.999	0.999
	Rao	0.205	0.570	0.860	0.969	0.999	1	1
50	K-S	0.877	0.987	0.999	1	1	1	1
	Rao	0.284	0.753	0.967	0.998	1	1	1
100	K-S	0.976	1	1	1	1	1	1
	Rao	0.486	0.953	0.999	1	1	1	1

the level significance of 5% using the tabulated critical values of the Kolmogorov-Smirnov statistic and the asymptotic critical values of the Rao's score statistic are indicated in Table 4 for some values of n and κ.
When the angle between the mean directions of the components of the mixture is 0°, the hypothesis H_1 reduces to the hypothesis H_0, and then the empirical power of the tests is equal to the size of the tests, which is indicated in Table 3. We have concluded the following:

- For the analysed cases, the Kolmogorov-Smirnov test is superior to the Rao's score test for small values of the concentration parameter, that is for values of $\kappa \leq 5$.
- For values of $\kappa > 5$, the two tests or have equal power or the Rao's score test has slightly greater power than the Kolmogorov-Smirnov test.
- As for the case of known parameters, we observed in this case that for fixed n, the empirical power of the tests increases or remains constant when the concentration parameter κ increases and for fixed κ, the empirical power of the tests increases when the dimension of the sample n increases.

Next, we consider the following alternative hypothesis:

H_1 : The sample comes from a Kent distribution $FB_5(\kappa, \beta, I)$, where I denotes the identity matrix.

Since we do not know the distribution of the test statistics under the alternative distribution, we have estimated the power of the tests by generating 100,000 samples from this distribution.

Table 5 Empirical power of the tests with an alternative of a Kent distribution

d	0.1				0.2			
n	30		50		30		50	
κ	K-S	Rao	K-S	Rao	K-S	Rao	K-S	Rao
3	0.143	0.548	0.147	0.652	0.162	0.864	0.182	0.959
5	0.207	0.424	0.193	0.526	0.247	0.806	0.249	0.933
10	0.225	0.335	0.214	0.414	0.321	0.742	0.343	0.890
20	0.209	0.286	0.188	0.354	0.299	0.692	0.306	0.857
30	0.204	0.276	0.189	0.346	0.295	0.683	0.312	0.853
50	0.198	0.256	0.173	0.318	0.280	0.663	0.279	0.833
d	0.3				0.4			
n	30		50		30		50	
κ	K-S	Rao	K-S	Rao	K-S	Rao	K-S	Rao
3	0.272	0.989	0.423	1	0.807	1	0.973	1
5	0.302	0.987	0.329	0.999	0.428	1	0.626	1
10	0.543	0.979	0.650	0.999	0.807	1	0.924	1
20	0.541	0.969	0.643	0.998	0.919	1	0.984	1
30	0.543	0.968	0.652	0.998	0.931	1	0.987	1
50	0.515	0.964	0.603	0.997	0.920	1	0.983	1

The empirical power of the tests obtained for a significance level of 5%, using the tabulated critical values of the Kolmogorov-Smirnov statistic and the asymptotic critical values of the Rao's score statistic is indicated in Table 5 for the sample sizes $n = 30, 50$ and some values of the parameters κ and d. When $d = 0$ or $\beta = 0$, the Kent distribution reduces to Fisher distribution given in the null hypothesis, and so in this case, the empirical power of the tests is equal to the size of the tests which is indicated in Table 3. We have obtained the following conclusions:

- For each sample size n, and fixed κ, the empirical power of both tests increases as the parameter β (or d) increases. This is as expected, because as β (or d) increases, the alternative hypothesis moves away far the null hypothesis.
- For fixed κ and β (or d), the empirical power of the two tests increases when the sample size n increases.
- For this type of alternative, the Rao's score test has empirical power greater than the Kolmogorov-Smirnov test. It is expected that the Rao's test is more powerful in this case as it is the likelihood ratio test of this H_0 against the Kent distribution.

6 Example

We have considered data consisting in 26 measurements of magnetic remanence in specimens of Palaeozoic red-beds from Argentina (Fisher et al. 1987, Appendix B2). We want to test

H_0 : The sample comes from a Fisher distribution $F(\boldsymbol{\mu}, \kappa)$.

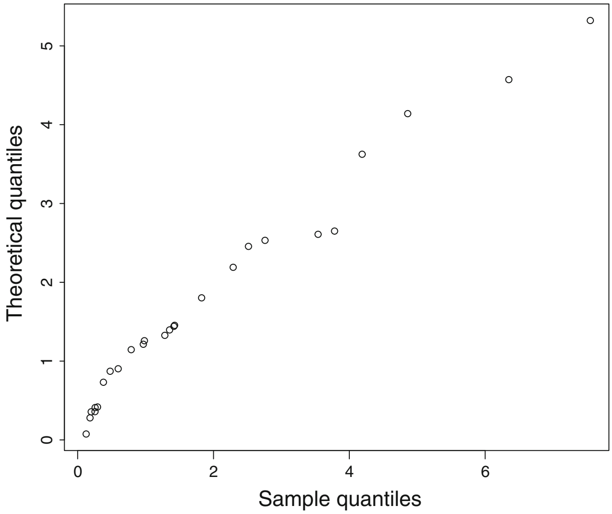

Fig. 2 Chi-square Q–Q plot

Then we reduce this hypothesis to the following:

H_0' : The values $y_i = 2\kappa\left(1 - \mathbf{x}_i'\boldsymbol{\mu}\right), i = 1, \ldots, 26$ are $\chi^2_{(2)}$ distributed for κ large.

Since the parameters $\boldsymbol{\mu}$ and κ of the Fisher distribution are unknown, we replace them by their maximum likelihood estimates, given by $\widehat{\boldsymbol{\mu}} = (-0.4394, -0.3169, -0.8406)'$ and $\widehat{\kappa} = 109$, for applying the goodness-of-fit techniques.

First we have obtained the chi-square Q–Q plot using R, which is indicated in Fig. 2. The goodness-of-fit of the $\chi^2_{(2)}$ distribution seems reasonable. Next we have applied the Kolmogorov-Smirnov test and we have obtained for the observed value of the Kolmogorov-Smirnov statistic: 0.111. Comparing this value with the tabulated critical values of the statistic given in Miller (1956), which are 0.311, 0.259 and 0.233, for the levels of significance of 1, 5 and 10%, respectively, we have concluded that the hypothesis of a $\chi^2_{(2)}$ distribution is not rejected for these levels.

We have also applied the the Rao's test. The observed value of the test statistic is 0.772 and comparing this value with the critical values of the statistic for the usual levels of significance of 1, 5 and 10% , equal to 9.21, 5.99 and 4.61, respectively, we do not reject the hypothesis of data to come from a $\chi^2_{(2)}$ population. Then, we can not conclude that these data do not come from a Fisher distribution.

7 Conclusion

We have suggested some goodness-of-fit methods for a concentrated von Mises-Fisher distribution, which had not yet been given in the literature. We have presented the empirical power of the Kolmogorov-Smirnov test for investigating the goodness-of-fit of a von Mises-Fisher distribution, for several dimensions of the sphere, supposing as alternative hypothesis a mixture of two von Mises-Fisher distributions with known parameters. We have also determined the empirical power of the Kolmogorov-Smirnov test for the fit of a Fisher distribution, supposing as alternative hypothesis a Kent distribution or a mixture of two Fisher distributions with parameters replaced by their maximum likelihood estimates and we have compared the power of this test with a goodness-of-fit test based on the Rao's score statistic. The simulation results revealed that the empirical power of these tests is good and increases rapidly as the sample size increases. When the alternative hypothesis is a mixture of two Fisher distributions, the Kolmogorov-Smirnov test is superior to the Rao's score test for small values of the concentration parameter (less or equal to 5), and for values of the concentration parameter greater than 5, the two tests are identical or the Rao's score test is slightly greater. When the alternative is a Kent distribution, the Rao's score test is superior to the Kolmogorov-Smirnov test. Finally, we have used an example of the literature to illustrate these methods.

Acknowledgments The author thanks the helpful comments and suggestions given by the referees of this journal.

Appendix

See Tables 6, 7, 8.

Table 6 Critical values of the Kolmogorov-Smirnov statistic for known parameters

p	n	10			20			30		
	$\kappa\backslash\alpha$	1%	5%	10%	1%	5%	10%	1%	5%	10%
2	3	0.495	0.416	0.375	0.360	0.302	0.271	0.299	0.250	0.224
	5	0.494	0.413	0.371	0.355	0.296	0.267	0.294	0.244	0.220
	10	0.491	0.410	0.369	0.356	0.295	0.266	0.291	0.241	0.218
	20	0.489	0.409	0.368	0.352	0.295	0.265	0.290	0.241	0.217
	30	0.490	0.409	0.369	0.354	0.294	0.265	0.288	0.241	0.217
3	3	0.487	0.409	0.368	0.353	0.294	0.264	0.291	0.241	0.218
	5	0.488	0.409	0.369	0.353	0.294	0.265	0.290	0.242	0.218
	10	0.488	0.409	0.369	0.353	0.294	0.264	0.290	0.242	0.218
	20	0.488	0.409	0.368	0.352	0.293	0.264	0.290	0.242	0.217
	30	0.488	0.409	0.368	0.353	0.295	0.265	0.289	0.242	0.218
4	3	0.499	0.419	0.378	0.374	0.313	0.282	0.316	0.266	0.239
	5	0.490	0.411	0.370	0.357	0.299	0.269	0.298	0.248	0.222
	10	0.486	0.407	0.367	0.351	0.295	0.265	0.291	0.242	0.218
	20	0.489	0.408	0.368	0.352	0.294	0.264	0.289	0.242	0.218
	30	0.489	0.410	0.368	0.351	0.294	0.265	0.289	0.242	0.217
10	30	0.502	0.422	0.383	0.368	0.312	0.282	0.317	0.265	0.238
	40	0.503	0.417	0.378	0.367	0.306	0.273	0.307	0.255	0.230
	50	0.492	0.412	0.370	0.360	0.301	0.270	0.299	0.250	0.225
	70	0.492	0.414	0.372	0.356	0.297	0.268	0.295	0.247	0.221
	100	0.490	0.411	0.370	0.352	0.295	0.266	0.292	0.244	0.219
Tabulated values		0.489	0.409	0.369	0.352	0.294	0.265	0.290	0.242	0.218

Table 7 Critical values of the Kolmogorov-Smirnov statistic for unknown parameters

p	n	20			30			50		
	$\kappa \backslash \alpha$	1%	5%	10%	1%	5%	10%	1%	5%	10%
2	3	0.385	0.341	0.318	0.345	0.310	0.291	0.308	0.280	0.265
	5	0.369	0.328	0.305	0.332	0.298	0.279	0.295	0.268	0.254
	10	0.361	0.321	0.299	0.324	0.290	0.272	0.288	0.261	0.247
	20	0.357	0.317	0.296	0.321	0.286	0.269	0.285	0.258	0.244
	30	0.357	0.315	0.294	0.319	0.286	0.268	0.284	0.257	0.243
3	3	0.376	0.284	0.246	0.309	0.234	0.202	0.238	0.180	0.156
	5	0.373	0.285	0.246	0.309	0.235	0.203	0.236	0.180	0.156
	10	0.381	0.290	0.250	0.312	0.238	0.206	0.243	0.185	0.159
	20	0.398	0.303	0.259	0.324	0.248	0.212	0.248	0.189	0.162
	30	0.387	0.294	0.253	0.313	0.241	0.208	0.241	0.188	0.163
4	3	0.385	0.341	0.318	0.345	0.310	0.291	0.216	0.180	0.164
	5	0.369	0.328	0.305	0.332	0.298	0.279	0.215	0.185	0.171
	10	0.344	0.265	0.234	0.273	0.218	0.197	0.212	0.177	0.161
	20	0.330	0.266	0.241	0.272	0.226	0.207	0.216	0.167	0.148
	30	0.348	0.262	0.229	0.282	0.215	0.190	0.218	0.169	0.149
10	20	0.325	0.254	0.226	0.269	0.215	0.193	0.218	0.188	0.175
	30	0.325	0.254	0.226	0.269	0.215	0.193	0.212	0.174	0.159
	40	0.350	0.255	0.223	0.265	0.209	0.187	0.203	0.164	0.148
	50	0.374	0.275	0.233	0.298	0.216	0.197	0.212	0.160	0.143
	70	0.311	0.255	0.229	0.261	0.213	0.192	0.206	0.169	0.153
Tabulated values		0.352	0.294	0.265	0.290	0.242	0.218	0.231	0.192	0.173

Table 8 Critical values of the Rao's score statistic ($p = 3$)

n	20			30			50		
$\kappa \backslash \alpha$	1%	5%	10%	1%	5%	10%	1%	5%	10%
5	8.791	6.030	4.771	8.960	6.077	4.751	9.114	6.073	4.673
7	8.483	5.913	4.705	8.803	5.998	4.690	8.943	6.015	4.651
10	8.269	5.850	4.664	8.595	5.908	4.646	8.898	5.963	4.632
20	8.121	5.758	4.604	8.566	5.871	4.614	8.800	5.893	4.611
30	8.104	5.712	4.566	8.435	5.821	4.596	8.816	5.902	4.597
40	8.021	5.731	4.561	8.498	5.838	4.584	8.838	5.911	4.599
50	8.046	5.734	4.592	8.467	5.828	4.580	8.695	5.879	4.598
$\chi^2_{(2)}$	9.210	5.992	4.605	9.210	5.992	4.605	9.210	5.992	4.605

References

Boulerice B, Ducharme GR (1997) Smooth tests of goodness-of-fit for directional and axial data. J Multivar Anal 60(1):154–175

Fisher NI, Best DJ (1984) Goodness-of-fit tests for Fisher's distribution on the sphere. Aust J Stat 26(2): 142–150

Fisher NI, Lewis T, Embleton BJJ (1987) Statistical analysis of spherical data. Cambridge University Press, Cambridge

Fisher NI (1993) Statistical analysis of circular data. Cambridge University Press, Cambridge
Kent JT (1982) The Fisher-Bingham distribution on the sphere. J R Stat Soc B 44(1):71–80
Lawson A (1988) Fitting the von Mises distribution using GLIM. J Appl Stat 15(2):255–260
Lewis T, Fisher NI (1982) Graphical methods for investigating the fit of a Fisher distribution to spherical data. Geophys J R Astr Soc 69:1–13
Lockhart RA, Stephens MA (1985) Tests of fit for the von Mises distribution. Biometrika 72(3):647–652
Mardia KV, Holmes D, Kent JT (1984) A goodness-of-fit test for the von Mises-Fisher distribution. J R Stat Soc B 51:111–121
Mardia KV, Jupp PE (2000) Directional statistics. Wiley, Chichester
Miller LH (1956) Table of percentage points of Kolmogorov statistics. J Am Stat Assoc Tables 72(3): 647–652 Sons, Chichester
Rivest L-P (1986) Modified Kent's statistics for testing goodness of fit for the Fisher distribution in small concentrated samples. Stat Prob Lett 4:1–4
Watson GS (1983) Statistics on spheres. Wiley, New York
Wood A (1994) Simulation of the von Mises-Fisher distribution. Commun Stat Simul Comput 23(1): 157–164