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Decision under risk as a multicriteria problem

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Abstract

Most of the approaches to decision problems under uncertainty are based on decision paradigms, generally associated to an optimization process that leads to a final solution. For the Decision Maker, the basic decision is thus what paradigm to choose, the rest of the procedure being mainly technical.

In this paper, a different approach is advocated for this kind of problems. The main idea is to leave prescriptive models in favor of a more flexible approach, where risk related criteria are explicitly considered, conducting to an "equivalent" multicriteria (deterministic) model where decision-aid procedures can be used, with a greater involvement of the Decision Maker.

The paper discusses first the uncertainty model and then reviews existing paradigms for the single criterion problem under uncertainty. Proposed risk and opportunity attributes come mainly from the analysis of those methodologies and from risk perception studies reports.

Some hints about multicriteria aid methods and an illustrative example complete the paper.

Keywords: Uncertainty modeling; Decision analysis; Multiple criteria analysis; Risk analysis; Fuzzy sets

1. Introduction

Decision under uncertainty is certainly one of the most frequent situations in practical decision-making, namely in planning activities in many fields. Modeling has been done mostly with scenarios, probabilistic or stochastic approaches and, in the last years, also with fuzzy set theory. Then, the usual approach for single criterion problems under uncertainty consists of selecting a decision paradigm, or rule, that finds the "optimal" solution of the problem. The expected value paradigm is by far the most frequent rule, but E-V analysis, utility theory, minimax approaches, regret minimization and fuzzy satisfaction have also been used as decision paradigms. Less frequently, minimizing risk is explicitly identified as a criterion, sometimes in a vague way, leading to different attributes and methodologies.

It is interesting to note that, when comparing the results produced by different paradigms, contradictory conclusions appear (this is particularly true between the expected value optimization and minimax, or regret minimization), which could lead to the meta-problem of choosing the "best" paradigm. However, our perspective is different – different

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paradigms may be considered as different points of view, equally interesting to characterize the uncertain outcome of an alternative, leading to a deterministic multicriteria problem. This is not completely new, so the aim of the paper is mainly to build a unified view over the issue and propose some procedures that may help selecting the indices that can be used as decision criteria.

The main advantage of this approach is to maintain the Decision Maker close to the process, which increases the robustness of the final solution regarding his preferences, even if a decision-aid phase is necessary to reach the preferred solution. We strongly believe that, in many circumstances, the insight gained by the Decision Maker more than compensates the potential disadvantages of increased complexity of the process. The systematic implementation of this concept leads to the identification of risk indices that can be used as attributes, but also of less frequently seen *opportunity indices* that may be useful to complete the picture.

The structure of this paper is the following. Probabilistic, fuzzy and scenario based models of uncertainty are addressed in Section 2, and decision rules are described in Section 3. Definitions of risk are invoked and compared (Section 4), in order to establish meaningful attributes, suitable for multicriteria analysis and decision-aid (Section 5). An illustrative example (Section 6), the conclusions and references complete the paper.

2. Uncertainty models

Dealing with uncertainty, or risk, includes building models for uncertainty (*what may happen*) and decision (*what to do*). Of course, in a less prescriptive way, we could substitute the latter by a decision-aid model (*how to help the decision maker*).

So, before going into decision-aid or decision-making models, it seems interesting to review the main approaches to model uncertainty *itself*. It is not important for the discussion if the problem is discrete, countable or continuous, so we will talk about alternatives, solutions or actions, always meaning possible decisions.

2.1. Scenarios

A convenient way of modeling uncertainty is the use of scenarios (e.g. Kouvelis and Yu, 1999). It is a ‘‘natural’’ approach, where the main uncertain variables are globally estimated, correlations are taken into account and different *structured futures* are constructed. Generally, it is possible to characterize each scenario by a linguistic label that shows the holistic nature of scenario definition.

The basic idea about scenarios does not require a probability distribution, so we will separate the two issues. However, the concept of ‘‘most likely scenario’’ is sometimes invoked in a qualitative way, which is a different thing – it means generally that all the other possible scenarios will be discarded.

Finally, scenarios are possible (future) instances of data and parameters, so an impact model is always necessary to turn this approach useful, by providing the means to evaluate the consequences of a decision in each possible scenario. The impact may be described by a single attribute (e.g. Cost) or multiple attributes. In this paper, only the single attribute situation will be discussed.

We may also have scenarios independently defined for several parameters (e.g. ‘‘three point estimates’’ of gas consumption – most likely, minimum and maximum). Conceptually, this is similar to the use of intervals or fuzzy sets.

2.2. Intervals

The other ‘‘natural’’ way of dealing with uncertainty is considering that some or all of the data and parameters are described by intervals, instead of a single real number. Intervals are also used to ‘‘translate’’ qualitative descriptions into numeric values.

In their basic formulation, intervals are not linked to probabilistic or possibilistic distributions – much like scenarios, they only try to capture every possible future value of the relevant data.

When only a parameter is at stake, the corresponding interval can be considered as an infinite set of scenarios. However, when more than one parameter is considered, independence issues arise, because some of the variations may be correlated. Therefore, independence assumptions conduct most of the times to unrealistic combination of extreme values.

2.3. Probabilistic models

In the framework of decision-making, we may say that probabilistic models are enhancements of the two previous approaches, when further

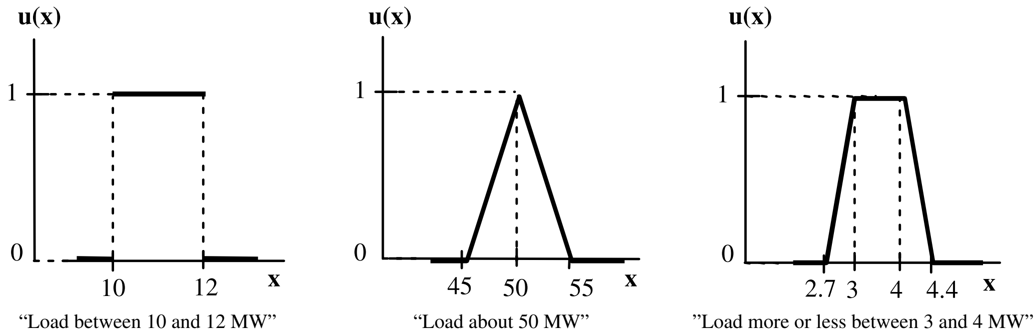

Fig. 1. Examples of fuzzy numbers.

information exists, coming from statistics or symmetry considerations.

Assigning probabilities to scenarios is a usual practice, although it is not uncommon that we face *subjective probabilities*, meaning a set of probability values estimated by an expert. Sometimes, these subjective probabilities are given as intervals, in what we could call a second-order uncertainty model. In this paper we will not address directly that issue, although most of the discussion can be applied to it.

A more frequent situation is to have probability distributions for the data, which allow us, in theory, to have a probability distribution of the consequences of each alternative. In the case of a single criterion, this means a probability distribution of the attribute.

2.4. Fuzzy models

Fuzzy set theory (e.g. Dubois and Prade, 1980, or Zimmermann, 1985) is another way of incorporating additional information in the uncertainty model, when no statistical information is available or when we are dealing with qualitative descriptions corresponding to expert declarations about the data or the impact of the alternatives.

Intervals are a special case of fuzzy values, corresponding to no additional information besides the range of possible values. On the other hand, a singleton fuzzy value signifies no uncertainty, that is, a real number.

The general fuzzy number can be seen as a set of nested intervals, with increasing degrees of membership (or possibility values). This means that each value of the support interval has assigned a real number (the degree of membership) that measures its compatibility with the vague declaration associated with the fuzzy number.

Popular models are triangular and trapezoidal fuzzy numbers (the latter includes the former as a special case). Examples of fuzzy numbers used in Power Systems models are shown in Fig. 1.

Similarly to the previous uncertainty models, an impact model is also needed in this case. Fuzzy numbers have their own operation rules but, for the purpose of this paper, it suffices to consider that it is possible to evaluate the impact of an alternative by a fuzzy number. Note however, in some cases, the fuzzy impact comes directly from an expert vague judgment, without resort to a mathematical impact model.

2.5. An unified view

It seems that a good point of departure in the single criterion problem under uncertainty is returning to a basic framework: each alternative has an outcome, described by

(a) a list of real numbers, when a finite number of scenarios exists;
(b) a list of pairs (attribute value, probability), when a finite number of scenarios exists with assigned probabilities;
(c) a discrete probability distribution or a probability density function, when the uncertainty is probabilistic;
(d) a possibility distribution, when a fuzzy model is used.

Note that, in the last two cases, a complete description needs the joint distribution for all the alternatives, since independence is not generally a correct assumption. This is the main difference between (b) and the discrete part of (c), because in the former case no ambiguity exists regarding the independence issue, due to the way scenarios are built.

To conclude this topic, let us point out that this distinction between the uncertainty model and the decision model is essential to clarify the assumptions and paradigms discussed in the next section.

3. Decision paradigms

We will begin by revising the main approaches, focusing in the concepts rather than in the mathematical details or algorithms, that can be seen in the references (besides, we are speaking about well-known methodologies). A common characteristic of the methodologies available to a decision maker for this situation is that they are generally prescriptive (with little exceptions), in the sense that, after being chosen, the paradigm goes to a "correct" or "optimal" decision with not too much intervention of the Decision Maker (Utility theory is a partial exception to this).

3.1. Expected value paradigm

Applied in probabilistic context, this paradigm may be described by the following decision rule:

Choose the alternative with the best expected value of the attribute.

Of course, "best" means "minimum" or "maximum" according to the type of problem. This paradigm assumes implicitly that a number of similar decision situations will be repeated over the time, although it is many times applied to isolated situations. Operationally, the rule is generally associated with the basic decision tree approach, where the tree is gradually reduced by the use of the mentioned decision rule (e.g. Clemen and Reilly, 2001).

The major criticism to this popular methodology relates to the fact that risk is ignored: tossing a coin between X and −X is always equivalent to receiving nothing, even if X is a million euro.

Independently of this discussion (not important to the purpose of this paper), we may say that the Expected Value Paradigm (EVP) is an aggregation procedure, that substitutes a probability distribution of outcomes by its expected value

$$A_k = \{(z_{ik}, p_{ik}), i = 1, \ldots, s\} \xrightarrow{\text{EVP}} A_k = \sum_{i=1}^{s} p_{ik} \cdot z_{ik},$$

$$A_k = f_k(z) \xrightarrow{\text{EVP}} A_k = \int_{-\infty}^{\infty} z \cdot f_k(z)\,\mathrm{d}z.$$

Therefore, the paradigm transforms the original problem in a deterministic single criterion problem.

3.2. Utility theory

Because of its mathematical elegance, utility theory is very attractive (Von Neumann and Morgenstern, 1944), but its practical application raises problems and inconsistencies, whose discussion is outside the scope of this paper (e.g. McCord and de Neufville, 1983). So, we will proceed by describing the features with interest to our discussion.

Briefly, we can state that the decision rule for this theory is the following: *Choose the alternative with the greatest expected utility*.

This is very similar to the decision rule defined by Bernoulli in the XVIII century (Bernoulli, 1738/1954), and also *seems* to be not very different from the previous rule. However, a great difference exists, because the utility function $u(z)$ can capture the decision maker attitude towards risk, as pictured in Fig. 2 for a maximization problem. Note that the "risk indifferent" function corresponds to the use of the expected value paradigm.

As mentioned before, we will not discuss here how to construct the utility function (or if it possible to do that in a reliable way), but it must be pointed out that the utility function is only instrumental for the purpose of comparing alternatives. Therefore, the utility values have no interest outside a specific decision problem and cannot be traded-off against attributes' values.

A curious thing about utility theory is that risk is not assessed explicitly (although intents in that direction exist, as discussed in the next section). So, we have here a fully normative procedure that, after the construction of the utility function,

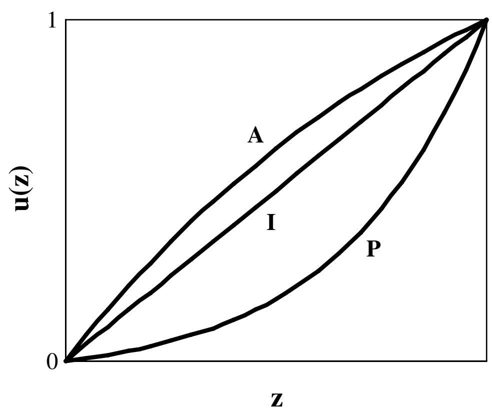

Fig. 2. Attitudes towards risk in utility theory (I – Indifferent, A – Risk Averse, P – Risk Prone).

chooses the best alternative without further interference of the decision maker.

Thus, we may want to use the methodology, if we accept the axioms and we are able to construct the utility function, but it is not easy to use it to build a criterion that reduces the complexity of the probabilistic description (although it must be mentioned here the *standard measure of risk* $R = -E[u(X - \bar{X})]$, proposed by Jia et al. (1999)).

3.3. E-V analysis and risk-based value functions

When trying to include risk as an attribute, variance is perhaps the first measure that comes to mind. Representation of alternatives in the E-V plan is common (for instance in portfolio theory), and corresponding prescriptive models with value functions have been developed.

We must point out that E-V representation does not imply necessarily the use of value functions. So, it may be used as a basis for non-prescriptive approaches, suitable for decision-aid, as we advocate in this paper. In Section 5 this idea will be developed.

Coming to value functions, we find (if z is a return attribute)

$$\max E[z] - \alpha \cdot E[(z - \bar{z})^2],$$

where $\alpha > 0$ is a risk aversion parameter ($\alpha < 0$ for risk proneness, $\alpha = 0$ for risk indifference). A less frequent alternative is

$$\max E[z] - \alpha \cdot E[(z - \bar{z})^2] + \beta \cdot E[(z - \bar{z})^3],$$

where $\beta > 0$ allows us to include the influence of the skewness of the outcomes in risk evaluation, with the idea that variance "in the good direction" means opportunity, not risk.

These approaches are alternative to utility theory models (although the two are sometimes confused), as they use value functions on deterministic indices, while utility functions are applied directly to the probability distributions of the attributes. In fact, the two formulas may be seen as approximations of the utility function. In that basis, however, they suffer some theoretical fragility (Dillon, 1971).

3.4. Robust optimization

An important line of thought regarding uncertainty problems is based on the concept of robustness and related notions, as disappointment or regret (e.g. Kouvelis and Yu, 1999). The basic idea corresponds to the well-known *minimax* paradigm: *Choose the alternative that, in the worst case, has the best value of the attribute.*

The name *minimax* comes from the mathematical formulation when minimizing costs under different scenarios:

$$\min_{z \in Z} \left\{ \max_{s \in S} Cost(z, s) \right\},$$

where Z is the set of alternatives and S is the set of scenarios.

This is obviously a conservative approach, suited for "one-shot" decision situations, i.e., when there is no place for repeated decisions where bad outcomes may be compensated by future good outcomes (for the discussion of this difference, see the interesting paper of Benartzi and Thaler, 1999).

Kouvelis and Yu (1999) classify the *minimax* principle as an *absolute robust* approach, suited for goal satisfaction or competition situations where the uncertainty comes from competitors' decisions (typically the most unpleasant to you). They make a distinction to the *minimax regret* approach, adequate to situations where the quality of the decisions are evaluated ex post facto, i.e., no one cares of the possible scenarios that finally did not happen, and market situations where your losses are automatically gains of your competitors.

Regret is defined as the difference between the outcome of the decision (after the resolution of the uncertainty) and the best possible outcome in the same scenario. Assuming again cost minimization, we will choose the alternative that corresponds to

$$\min_{z \in Z} \max_{s \in S} Regret(z,s) = \min_{z \in Z} \max_{s \in S} (Cost(z,s) - Cost^*(s)),$$

where $Cost^*(s) = \min_{z \in Z}(Cost(z, s))$ is the best possible outcome in scenario s.

These basic formulations ignore any existing additional information, like probability distributions. Still, other approaches include probabilities in the definition of regret. For instance, in a power distribution system planning approach, Miranda and Proença (1998) used

$$Regret(z, s) = p(s) \cdot (Cost(z, s) - Cost^*(s)).$$

On the other hand, note that the robustness concept is not restricted to scenario modeling and may be used with intervals or fuzzy sets (in the latter case, with a regret definition that includes possibilities).

An interesting concept related to regret is that of *exposure*, defined as the percentage of scenarios where alternative z conducts to acceptable regret (defined by a threshold)

$$Exposure(z) = \frac{\#\{s \in S | Regret(z,s) \leqslant threshold\}}{\#S}.$$

Robust optimization is a prescriptive approach that aims at obtaining an optimal solution. However, formal definitions of regret and exposure are important contributions that can be used in decision-aid approaches (see Section 4).

3.5. Bellman and Zadeh fuzzy decision

When uncertainty is modeled with fuzzy sets, the first paradigm that comes into operation is due to Bellman and Zadeh (Dubois and Prade, 1980). It is a symmetrical approach, in the sense that satisfaction of constraints and goals is unified, so each alternative z has a degree of membership to the Fuzzy Decision ($\tilde{D}$) equal to the minimum of the degrees of membership to the fuzzy sets Goal ($\tilde{G}$) and Restriction ($\tilde{R}$):

$$u_{\tilde{D}}(z) = \min\{u_{\tilde{G}}(z), u_{\tilde{R}}(z)\}.$$

In a more meaningful way, we could say that $u_{\tilde{R}}$ is the degree of feasibility, while $u_{\tilde{G}}$ measures the degree of satisfaction of the goal. Fig. 3 shows the three membership functions in an example where we want to choose a number that is

$\tilde{G}$ – clearly greater than 4,
$\tilde{R}$ – around 5,

resulting in the fuzzy decision $\tilde{D}$.

The final step of the process is the selection of an alternative. In this framework, it is natural that the alternative with the maximum mode of $\tilde{D}$ (the value x with the greatest $u_{\tilde{D}}(x)$) will be selected ($x = 6$ in the example). Intervention of the Decision Maker in this approach is restricted to the definition of the goal, a decisive parameter of $u_{\tilde{G}}$, and definition of the tolerances that complete the definition of $u_{\tilde{G}}$ and $u_{\tilde{R}}$.

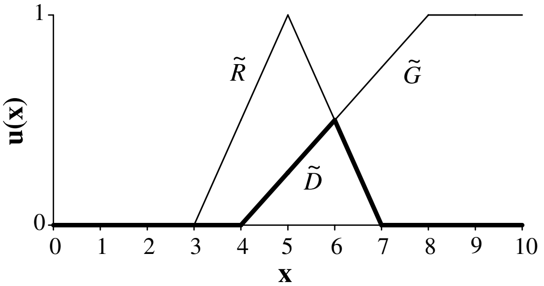

Fig. 3. Fuzzy decision (Bellman and Zadeh).

This framework for decision in a fuzzy environment is the support for many approaches, namely for fuzzy linear programming (Lai and Hwang, 1992). Some of the models are softer than the original idea, allowing interactive intervention of the Decision Maker, that may alter goal definition or choose the acceptable level of unfeasibility ($1 - u_{\tilde{R}}$) during the process, after being informed of the consequences of preliminary decisions. In some cases, symmetry between goals and constraints is thus abandoned, but the general idea is maintained. See also Czogala and Zimmermann (1986) for a link with Section 3.3.

3.6. Other approaches

Of course, many approaches were left outside the previous review, although we think that the most important ones were addressed. To conclude this topic, we still want to mention two *incorrect* approaches:

(a) The *equal likelihood principle*, that considers that, if you do not know the probabilities, you set them all equal, so the decision rule consists on summing up the outcomes in all the scenarios and choosing the best aggregated value. Of course, this could be done if we estimate that the probability values of the different scenarios are not so different – it would then be an application of the expected value paradigm. However, this principle is most of the times presented (and sometimes applied) in situations where there is a complete lack of information about the likelihood of future scenarios, which means that the probability values are completely arbitrary. Therefore, the decision that comes from the application of the principle may be misleading to the Decision Maker, if he is not aware of this arbitrariness. Seeking for more information or using robust analysis seems to be more appropriate in this situation.
(b) *Ordering rule*: choose the alternative with the best sum of classifications in the scenarios. This approach is even worse than the previous one, since it does not take into account

the differences between the outcomes of the different alternatives in each scenario (only the order), while it also uses arbitrary probabilities.

4. Risk indices

From the previous discussion about methodologies, the concept of *risk* emerges as an important way of qualifying possible decisions. In fact, it is probably the most frequent word when people express their concerns about deciding under uncertainty.

According to Anders et al. (1999), "risk is associated with the lack of certainty of an outcome and how sensitive one is to that outcome and thus to the uncertainty". Of course, other definitions could be given, but the main point is that risk is a somehow vague concept, that needs further elaboration to become operational.

For the purpose of this paper, we are only interested in measures of risk, or risk indices. Regarding details of risk theory, see Jia et al. (1999) for a review on perceived risk models, Fishburn (1984) for an example of an axiomatic approach, or Bell (1983, 1995).

4.1. Outcome related indices

Reviewing the literature, we see that risk indices are generally (but not always) related with parameters of the probability distribution of the outcomes:

- Variance (or Standard deviation).
- Skewness.
- Standard difference.
- Probability of a negative outcome.
- Variance of negative outcomes.
- Expected value of losses.
- Worst-case value.
- Regret.
- Exposure.

We note the most of the indices are related to the belief (supported by psychological studies, e.g. Fischer et al., 1986) that people evaluate differently negative and positive outcomes, as compared to a benchmark or, in general, to the expected value of the outcomes. The three last indices were discussed previously in this paper and are independent of the uncertainty model.

4.2. Constraint related indices

A different kind of indices is related to *model robustness* (Kouvelis and Yu, 1999), when uncertainty is related mainly to constraints. So, these indices are intended to aggregate the degree (or probability) of violation of the constraints:

- *Robustness* – used implicitly in stochastic programming (Yeh, 1985) and in fuzzy linear programming, it corresponds to the maximum degree (or probability) of violation in all the constraints. Sometimes, its counterpart (1-Robustness) is called exposure, which may lead to confusion with the concept mentioned in the previous section.
- *Severity* – an index proposed by Matos and Ponce de Leão (1995) in planning problems with fuzzy constraints, to complement the concept of robustness. It tries to capture the intensity of violation of fuzzy constraints like $\tilde{b}_k(x) \leqslant b_k^{\mathrm{LIM}}$ by a possible solution z whose impact in constraint k

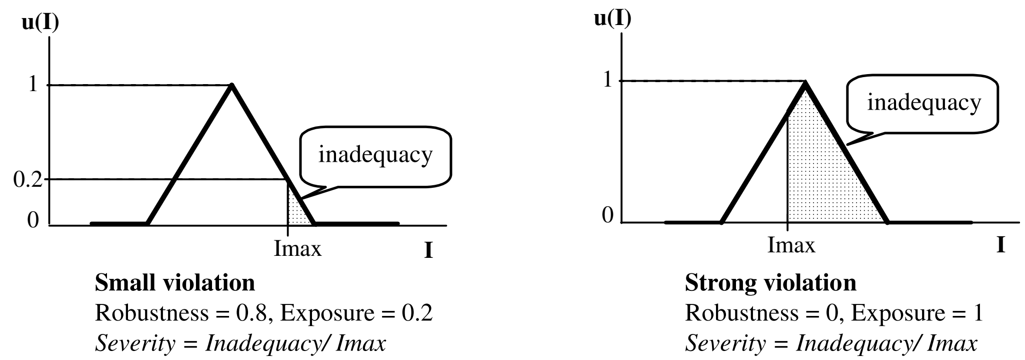

Fig. 4. Two examples of fuzzy constraints.

is described by the membership function $u_z(b_k) = u_{\tilde{b}_k(z)}(b_k)$. If $b_k^{\max}$ is the supremum of b_k such that $u_z(b_k) > 0$, the overall severity of z for the set C of all the constraints is given by

$$Sev(z) = \sum_{k \in C} \frac{\max\left\{0; \int_{b_k^{\mathrm{LIM}}}^{b_k^{\max}} u_z(b_k)\right\}}{b_k^{\mathrm{LIM}}}.$$

- *Inadequacy* – corresponds to the numerator of the preceding fraction (Proença, 1997).

Fig. 4 illustrates two cases of the calculation of the three indices to a constraint on fuzzy branch loads in an electric distribution system. Note that the first two indices have probabilistic counterparts, respectively the probability of violation and the sum of expected violations. In chance-constrained programming (Yeh, 1985), for instance, the Decision Maker sets a limit for the probability of violation of the constraints.

5. A multicriteria approach

As stressed before, the approaches to optimization under uncertainty are typically prescriptive. After a "meta-decision" (which paradigm to use?) a final solution is, in most of the cases, immediately obtained, without any further intervention of the Decision Maker.

However, most of the times different aggregated attributes to evaluate the alternatives are identified, some related with central tendencies, others with risk.

We advocate the substitution of the original description of the alternatives by a set of this kind of attributes, taken from different methods and risk definitions, in order to build a multicriteria characterization of the alternatives. On the other hand, we believe that a meaningful description of the impact of an alternative should not concentrate only in central measures and risk indices, but also in *opportunity indices* that reflect some good outcomes of the decision.

This approach is supported by investigations on the perception of risk (Benartzi and Thaler, 1999) that showed a considerable difference in the decision attitude when the actual possible consequences of the decision are presented, instead of a vague, global description. However, the cognitive limits of human perception do not favor a complete description, so something about seven percentiles (Miller, 1956) could be the limit.

So, we are not offering one more paradigm, but just trying to preserve information for the Decision Maker and letting him use multicriteria-aid tools to find his preferred solution.

The multicriteria model must be obviously related to the type of uncertainty, so we will consider three situations:

(a) Probability models (including scenarios characterized by a probability).
(b) Fuzzy models (which includes intervals as a particular case).
(c) Pure scenarios (when no probabilities are known).

5.1. Probability models

5.1.1. Central measure attribute

In this case, the expected value of the original attribute is the natural central measure. However, the mode could also be used in some cases.

5.1.2. Risk and opportunity indices

To complement the information given by the expected value, risk indices such as the ones mentioned in Section 4 can be used. If we note, however, that most of the traditional risk indices have a conservative tendency that limits the overall representation of the outcome, it is interesting to complement the list with opportunity indices:

- Probability of a positive outcome.
- Variance of positive outcomes.
- Expected value of gains.
- Best-case value.

The fact that people are more sensitive to losses than to gains does not affect the interest of considering also one or some of these indices, since we are not constructing any prescriptive value function but only identifying the relevant points of view for the multicriteria approach.

In the same direction, the Decision Maker may want to describe the outcome by a set of percentiles, defined in one of the following ways:

- *Probability defined* – a set of values of z corresponding to predefined probabilities of not being exceeded (in the case of scenarios, this would be the natural choice, since the probabilities are already defined). For instance, if z is a cost, we may define a risk index z_1 (cost not being exceeded 90% of the times) and an opportunity index z_2 (cost not being exceeded at least 10% of the times). This defines a sort of range of the consequences of the decision.

$$\{z_1, z_2 | p(z \leqslant z_1) = 0.9, p(z \leqslant z_2) = 0.1\}.$$

- *Value defined* – a set of probabilities corresponding to predefined levels of satisfaction of the attribute. For instance, if z is a net return, we may define two risk indices (probability of having losses and probability of a net return not exceeding 100), and an opportunity index (probability of a net return greater than 1000). Of course, 100 and 1000 would be anchor values stipulated by the Decision Maker, during the decision-aid process.

$$\text{Risk indices}: \{p_1 = p(z \leqslant 0), p_2 = p(z \leqslant 100)\},$$

$$\text{Opportunity index}: \{p_3 = p(z \geqslant 1000)\}.$$

5.1.3. Constraint related attributes

In this case, a natural attribute of a solution z related to each "probabilistic" constraint is the probability of violation of the constraint. When dealing with many constraints (the usual situation) a global attribute may be constructed using the *max* operator (a robust approach) or calculating the global probability of violation

$$p = p\left\{\bigcup_{k \in C} (b_k(z) > b_k)\right\},$$

where C is the set of constraints and b_k is the RHS term of a less than or equal constraint. Note that the calculation must take into account independence issues. If the Decision Maker wants also to see a measure of the severity of the violation, the expected value of the violations may be calculated and presented.

5.2. Fuzzy models

When the uncertainty model leads to a fuzzy (or interval) attribute, there are some differences in the construction of the multicriteria model, although the basic philosophy is the same.

5.2.1. Central measure attribute

First, the question of a reference value, to do the same job the expected value does in probabilistic models. This process is known in fuzzy set theory (namely in fuzzy control applications) as *defuzzification*, and can be done in a number of ways (see Zimmermann (1987) for a discussion on this topic):

(a) Center of gravity.
(b) Removal.
(c) Maximum mode.

The center of gravity is the fuzzy counterpart to the expected value of probability distributions. For a fuzzy outcome $\tilde{z}$ defined by a membership function $u(z)$ in the interval $[a \ldots b]$, the center of gravity is given by

$$z_0 = \frac{\int_a^b z \cdot u(z)\,\mathrm{d}z}{\int_a^b u(z)\,\mathrm{d}z}.$$

For an interval, we use the formula with $u(z) = 1$, $\forall z \in [a \ldots b]$, so $z_0 = (a + b)/2$.

Removal is one of the oldest (in fuzzy set theory) methods adopted to compare fuzzy quantities (Kaufmann and Gupta, 1988). In its general form, the removal is defined with respect to a reference point z_k, as shown in Fig. 5 ($z_k = 2$) for a Triangular Fuzzy Number (TFN), but most of the times $z_k = 0$. In this latter formulation, the removal may be interpreted as an approximation of the center of gravity, easier to calculate. On the other hand, for fuzzy numbers that are symmetric, the removal coincides with the center of gravity.

The maximum mode is sometimes obvious (triangular numbers) or corresponds to the average of the points with $u(z) = 1$ (intervals or trapezoidal numbers).

5.2.2. Risk and opportunity indices

Some of the indices mentioned previously can be used as attributes in fuzzy models, so we will not

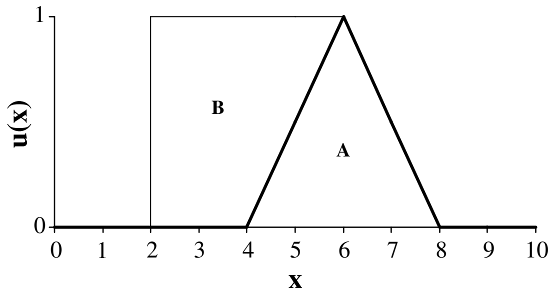

Fig. 5. Calculation of the removal. Removal (TFN, 2) = (2 · B + A)/2.

repeat them and list here only specific indices, some adequate to measure risk, others to measure opportunity:

(a) Largest and smallest of maximum.
(b) Largest and smallest possible values.
(c) Divergence.
(d) Measures of fuzziness.
(e) Set of possible results.

Case (a) corresponds to take only the left (smallest) or right (largest) of the values of the attribute with $u(z) = 1$. In case (b) we have $\inf(z|u(z) > 0)$ (smallest) and $\sup(z|u(z) > 0)$ (largest), corresponding roughly to the worst-case and best-case situations mentioned before (maximization). Divergence is simply the difference between the two preceding values (Kaufmann and Gupta, 1988).

According to Dubois and Prade (1980), a measure of fuzziness is "a scalar index to measure the degree of fuzziness of a fuzzy set". Many measures were proposed by different authors (see Klir and Folger, 1988) and a discussion on the topic would be beyond the scope of this paper, so we will give only an example of such a measure, based on the concept of entropy

$$f(A) = \sum_{x \in X} (u_A(x) \cdot \log_2 u_A(x) + (1 - u_A(x)) \cdot \log_2(1 - u_A(x))).$$

Of course, meaningfulness of the selected measure must be a concern in a decision-aid process.

Finally, case (e) corresponds to the percentile proposal of Section 5.1.2, that is, the selection of a limited set of results able to characterize the outcome of an alternative. Again, we have two hypotheses

- Best (or worst) values for a set of predefined α-cuts, through $\inf(z|u(z) > \alpha)$ and $\sup(z|u(z) > \alpha)$;
- Maximum degree of membership of a set of predefined values.

5.2.3. Constraint related attributes

Indices like the ones described in Section 4.2 may be used.

5.3. Pure scenarios

In the case of scenarios not characterized by probabilities (or possibilities) it is not possible to apply most of the indices mentioned in previous sections. So, the following hypotheses remain:

- Direct use of the scenarios' outcomes (see the considerations in the introduction to Section 5).
- Regret.
- Exposure.
- Minimax value.
- Maximax value – outcome of the alternative in the most favorable scenario.

Since the first three indices were already discussed, one word only to the last one, generally considered as too optimistic when used alone, but suitable for a multicriteria approach, as an opportunity index.

5.4. Multicriteria model and first stage of decision-aid

The preceding sections showed a list of deterministic attributes that can be used to describe the uncertain outcome of an alternative.

The next step is in theory the same of any multicriteria approach: help the Decision Maker select the relevant attributes, using the general principles of decision-aid (e.g. Roy, 1985). Note that all the discussion was independent of the nature of the problem, i.e., we will use the same general strategy for problems coming from mathematical programming and for problems with a limited set of alternatives. In the first case, the result will be a deterministic multi-objective problem; in the second, a multiattribute problem will be constructed.

Due to the substantial number of possible risk and opportunity indices that can be used, building the multicriteria model may benefit of some guidelines, and also from some measures of discriminative power of the indices. We will address now these two issues.

5.4.1. General guidelines

In order to build a meaningful representation of the impact of the alternatives, we recommend that, as a starting point, at least three criteria should be used: a central measure, a risk index and an opportunity index.

Besides the consideration of "mathematical" indices (e.g. variance, skewness), definition of the risk and opportunity indices should be based on important goals, anchor points, or soft thresholds, that the Decision Maker has in mind, like "maximum acceptable loss", "desirable return", "catastrophic result", "very good result", etc.

On the other hand, probability related indices meaningful to the Decision Maker should also be identified during the decision-aid process, sometimes in relation with vague descriptions of the consequences, like "probability of a small loss", "probability of a big loss" (risk indices) or "probability of an excellent return" (opportunity index). This kind of information could be used to build fuzzy indices or constraints, but it may happen that the DM wants to define the limit for "small loss", "excellent return", etc., leading either to ordinary constraints or to indices associated to the probabilities.

Finally, constraint related indices can be added to the list, if the problem includes probabilistic or fuzzy definitions of the feasibility of an alternative.

This interaction with the Decision Maker is essential to set a number of candidate indices, but may produce an excessive number of them, due to hesitations, ambiguities, etc. Therefore, in order to maintain the number of criteria of the multicriteria model within acceptable limits, pruning is necessary.

5.4.2. Using entropy as a measure of discriminative power

In order to support the process of choosing the final set of indices, we suggest the use of an entropy measure proposed by Zeleny (1982) in a different context. The idea is to characterize each candidate attribute k regarding its information content, so the candidates with a lesser entropy value should in principle be preferred.

Following the procedure, we first normalize the values a_{ik} of the outcomes of each alternative i ($i = 1, \ldots, N$) in criterion k:

$$b_{ik} = \frac{a_{ik}}{\sum_i a_{ik}}, \quad \text{all } i.$$

Next, we calculate the associated normalized entropy for criterion k, through:

$$e_k = \frac{-\sum_i b_{ik} \cdot \log_2 b_{ik}}{\log_2 N}.$$

This index takes the value 1 when all the performances are equal (no discriminative power) and goes to 0 when the information content increases (and so the discriminative power of the criterion). Note that, if $b_{ik} \to 0$, then $b_{ik} \cdot \log_2 b_{ik} \to 0$.

An alternative way of using the entropy measure consists in applying the procedure on the *differences* of the alternatives' performances relative to some threshold (e.g. the minimum cost). This may be adequate when there is some common quantity affecting all the alternatives, increasing the value of e_k. On the other hand, if some of the performances are negative, this kind of transformation is mandatory.

In any case, the entropy measure is just an auxiliary tool for the pruning process – the global meaningfulness of the attribute to the Decision Maker should conduct the process.

5.5. The second stage of decision-aid

Finally, there is the final decision-aid phase. As pointed out in the introduction, the aim of this paper is the construction of the multicriteria model for the single criterion problem under uncertainty, so we will not elaborate over this issue, but we may say that any procedure would be acceptable, since the history of the process is not important: we just face a multicriteria problem, with attributes meaningful to the Decision Maker, because they were built and selected in strong interaction with him. Possible methodologies include:

(a) Multiattribute value functions properly constructed (e.g. Keeney and Raiffa, 1976), which is not the case of some existing functions (cf. Section 3.3), defined with little incorporation of the DM preferences.
(b) Binary comparison methodologies like MACBETH (Bana e Costa and Vansnick, 1999), that uses qualitative declarations about the differences of preference between alternatives in each criterion to build an additive value function.
(c) Outranking methods (e.g. Roy and Bouyssou, 1993), that incorporate situations of indifference and hesitation between indifference and preference and accept veto declarations, building a partial preorder of the alternatives that can be afterwards be exploited.
(d) Interactive procedures (in the case of multiobjective problems), e.g. Teghem and Kunsch (1984), that already include some of the ideas presented in this paper.
(e) Non-parametric interactive procedures for multiattribute problems with many alternatives (e.g. Matos and Miranda, 1988) that just help the Decision Maker navigating through the set of alternatives.
(f) Methods based on fuzzy inference systems (e.g. Matos, 2004), that try to capture the

complexity of the Decision Maker's *value assessment* through fuzzy concepts and rules.

6. An illustrative example

We will give now a sketch of the ideas presented in the paper, by means of the example presented in Table 1, where the cost of 10 alternatives in three scenarios C1, C2 and C3 is presented. The table includes already a central value measure (expected cost) and five possible indices that may be used as attributes in the decision process: the worst cost of each alternative, the standard deviation of the cost, the probability of getting a cost over 60 (an important reference value for the Decision Maker), the skewness of the distribution (where negative values indicate an asymmetry in favor of low costs) and the maximum cost that may occur with a probability greater than 70%. In this case, no opportunity indices were defined, but the process would be similar. The table is completed with the normalized entropy of each candidate criterion, calculated for the difference regarding the minimum value.

Now, instead of choosing one decision paradigm, we consider some or all of these attributes, according to their meaningfulness for the Decision Maker – in short, we build a deterministic multicriteria model for the problem that substitutes the original problem.

Since the number of attributes is high, we may use the entropy values to help select the indices with greater discriminative power – in this case the expected value, the probability of a cost greater than or equal 60 and the maximum cost that may occur with probability greater than 70%. However, as we stressed before, this is only an auxiliary measure, that does not substitute the Decision Maker.

Assuming that, in this particular case, the Decision Maker wants to see the result of considering only the first two points of view, the original single criterion problem with uncertainty is transformed in the deterministic bicriteria problem depicted in Fig. 6.

In this framework, it is easy to see that only alternatives 2, 5 and 6 are nondominated. In a small problem like this one, the DM would probably have no difficulty in making the final decision, but of course decision-aid could be provided in a more complex problem.

On the other hand, in order to see the limitations of the Standard Deviation as a risk index, we used it instead of the worst cost as the second attribute, as shown in Fig. 7. The number of nondominated solutions increases, with the addition of alternatives 4 and 7 to the previous ones $(2, 5, 6)$. However, since the expected value of the cost of these alternatives

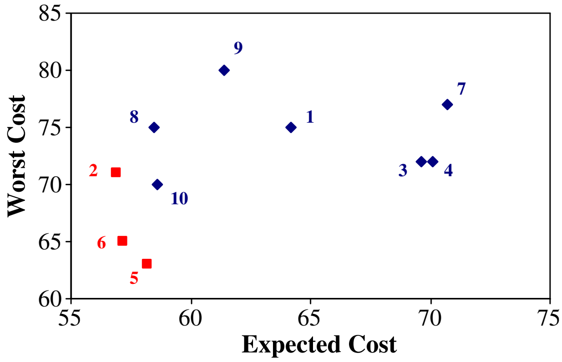

Fig. 6. Alternatives in the attribute space (Expected and Worst cost).

Table 1
Cost of 10 alternatives in 3 scenarios

n	C1 (0.3)	C2 (0.6)	C3 (0.1)	Expected cost	Worst cost	Standard dev	Prob. cost > 60	Skewness	Max cost prob ⩾ 0.7
1	59	65	75	64.2	75	4.49	0.7	0.93	65
2	50	58	71	56.9	71	5.91	0.1	0.89	58
3	68	72	60	69.6	72	3.67	1.0	−1.65	68
4	69	72	62	70.1	72	3.01	1.0	−1.80	69
5	53	60	63	58.2	63	3.52	0.7	−0.64	60
6	51	59	65	57.2	65	4.42	0.1	−0.24	59
7	68	71	77	70.7	77	2.49	1.0	1.24	71
8	56	57	75	58.5	75	5.52	0.1	2.63	57
9	62	58	80	61.4	80	6.45	0.4	2.31	58
10	62	55	70	58.6	70	4.92	0.4	1.11	55
	Normalized entropy			0.7851	0.9122	0.8955	0.8095	0.8835	0.8571

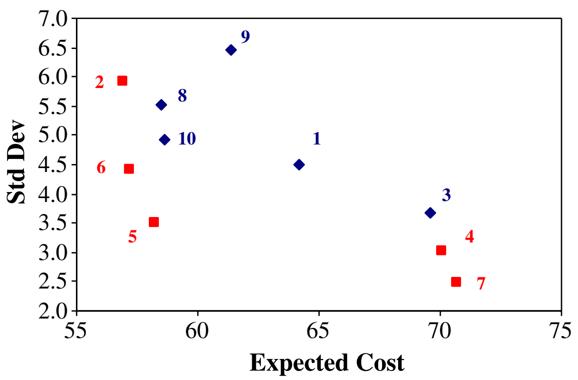

Fig. 7. Alternatives in the attribute space (Expected cost and Std Dev).

is high, the fact that the "risk" is small is not very encouraging. Something similar could be said about skewness.

On the other hand, if the first and fourth attributes were chosen, following the advice of the entropy measure, alternative 2 would dominate all the others, which is of course a not frequent situation but shows how far the reduction of the initial solution set can go.

Finally, if the Decision Maker wants to include many points of view (e.g. the three attributes with lower entropy value, plus the worst cost attribute), it would be advisable to use a decision-aid procedure, like for instance an outranking method, since there are 5 nondominated solutions (2, 5, 6, 8, 10). The application of such a method is, of course, outside the scope of this paper.

7. Conclusions

When dealing with uncertainty, the use of predefined decision paradigms is reassuring for the Decision Maker, but may conduct to undesirable solutions, due to its excessive prescriptive nature, that leaves not much room for the Decision Maker to express his preferences. On the other hand, since paradigms are generally presented as *the* paradigm, we are not always sure that the Decision Maker is aware of the possibility of considering different points of view.

However, analysis of existing methodologies to deal with uncertainty in single criterion problems allowed us to identify a number of deterministic attributes, suitable to represent, in a decision-oriented way, the uncertain outcome of an alternative, irrespectively of the uncertainty model behind the methodology (probabilities, scenarios, fuzzy sets).

We also advocated that a correct deterministic representation of the uncertain outcomes of the alternatives should include opportunity indices – the "optimistic" counterpart of risk indices – in order to avoid skewing the perspective, independently of the risk attitude of the Decision Maker.

The basis for a multicriteria approach was presented, aiming at changing from a prescriptive to a decision-aid paradigm. Guidelines for the model construction phase, with strong interaction with the Decision Maker, and an auxiliary entropy measure of the discriminative power of an index were discussed.

The example illustrates the main idea that this transformation of a decision problem under uncertainty in a deterministic multicriteria problem provides more meaningful information to the Decision Maker, and simultaneously may reduce significantly the dimension of the problem at hand. Moreover, the Decision Maker is not constrained by a predefined paradigm that in most cases does not capture all his concerns.

A final word about multicriteria problems under uncertainty. Although they were not addressed in the paper, all the considerations, procedures and examples can be extended to address them. The construction phase, of course, would be more demanding, since it necessary to safeguard all the original points of view while building new ones, but the essence of the methodology does not change.

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