Classifying C^{1+} Structures on Dynamical Fractals: 1 The Moduli Space of Solenoid Functions for Markov Maps on Train Tracks.

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Abstract

Sullivan's scaling function provides a complete description of the smooth conjugacy classes of cookie-cutters. However, for smooth conjugacy classes of Markov maps on a train track, such as expanding circle maps and train track mappings induced by pseudo-Anosov systems, the generalisation of the scaling function suffers from a deficiency. It is difficult to characterise the structure of the set of those scaling functions which correspond to smooth mappings. In this paper we introduce a new invariant for Markov maps called the *solenoid function*. We prove that for any prescribed topological structure, there is a one-to-one correspondence between smooth conjugacy classes of smooth Markov maps and pseudo-Hölder solenoid functions. This gives a characterisation of the moduli space for smooth conjugacy classes of smooth Markov maps. For smooth expanding maps of the circle with degree d this moduli space is the space of Hölder continuous functions on the space $\{0, \ldots, d-1\}^{\mathbf{N}}$ satisfying the matching condition.

1 Introduction.

In this paper we consider Markov maps on train tracks. These are both interesting in their own right and also arise naturally in the study of many dynamical problems. For example, one can associate a pair of these to any pseudo-Anosov diffeomorphism of a surface. The C^k structure of the pseudo-Anosov system f determines a C^{1+} structure on each of the train tracks in which the Markov maps are smooth and these determine the C^k conjugacy class of f. In [5] we explain the associated correspondence between families of pairs of solenoid functions and the smooth classification of families of maps between 2-dimensional foliated manifolds such as pseudo-Anosov systems. Other examples of Markov maps on train tracks include expanding maps of the circle and expanding maps on 1-dimensional branched manifolds. There are also interesting applications of

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the related Markov families (see [9] and [10]) to the study of the return maps of the Teichmüller flow on sections of the moduli space (see [1], [2], [5]) and also to the problems associated with convergence of renormalisation. Applications to infinitely renormalisable unimodal maps, bimodal maps and homeomorphisms of the circle are described, for example, in [6], [9], [10] and [11]. Moreover, in [4], we use Markov families (horocycle families) to generate the moduli space of C^{1+} smooth diffeomorphisms of the circle with rotation number of constant type. Some other examples are described in detail later in this section and in sections 1.1 and 5.

In section 4 we define the *scaling function* for a Markov map on a train track. This is an obvious extension of Sullivan's scaling function for cookie-cutters on binary Cantor sets ([13]). In section 10 we prove that the scaling function is a complete invariant for smooth conjugacy classes of Markov maps. However, it suffers from the serious deficiency that there is not a one-to-one correspondence between smooth conjugacy classes of Markov maps and scaling functions. This is because for general Markov maps such as expanding maps of the circle it is difficult to determine which scaling functions actually arise. For example, in subsection 4.1 we give some simple examples of scaling functions without corresponding smooth Markov maps. This raises the open problem of finding a better complete invariant.

In section 5, we solve this by introducing a new concept: the *solenoid function*. Whereas the scaling functions compare scales at different levels in the cylinder structure, the solenoid function compares them at the same level. It is more democratic. The reason for the name is that solenoid functions are in one-to-one correspondence with transversally continuous affine solenoid laminations (see [3], [16] and [17]).

In section 9 we construct from a solenoid function the canonical set $\mathcal{C}$ of smooth charts for the domain of a Markov map. These give a train track with a canonical smooth structure and a Markov map which is smooth in this structure. In section 12 and 13 we prove the one-to-one correspondence between smooth conjugacy classes of smooth Markov maps and solenoid functions.

These results concern C^{1+} smoothness. We also address the corresponding questions for $C^{1+\alpha}$ smoothness for a specific value of α. We define an α-*solenoid function* in section 6. In section 14 we prove that the space of the α-solenoid functions is a moduli space for the $C^{1+\alpha^-}$ smooth conjugacy classes of $C^{1+\alpha^-}$ smooth Markov maps. If the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is not an η-solenoid function for some $\eta > \alpha > 0$ then the Markov map F is not $C^{1+\eta}$ smooth.

In section 7 we define the properties of *determination* and α-*determination* for solenoid functions $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ of the Markov map F. Solenoid functions with either of these properies are said to be *determined* or α-*determined*. We prove that if the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is determined then there is a smooth Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$. The proof is in section 14. If the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is α-*determined* then the Markov map G is $C^{1+\alpha^-}$ smooth.

In section 8 we give a balanced equivalence between the geometry of the Markov partitions and smoothness for Markov maps. We prove the theorems of section 8 in section 11.

The results in this paper extend to Markov families (see [8] and [9]).

Many of the results that we prove here are corollaries of the results of the companion paper [7]. Some readers may feel that logic dictates that [7] should be read before this paper. However, we have chosen to stress the applications of [7] and have therefore made this paper the first in the series.

1.1 Some examples.

Before proceeding with the statements of our results we consider some simple examples to introduce and motivate our concepts and results.

1.1.1 Expanding circle maps.

An expanding circle map $E : S^1 \to S^1$ is a C^{1+} map with the property that in some smooth metric $|dE(x)| > \lambda > 1$, for all $x \in S^1$. For simplicity, let us restrict to the orientation preserving case. According to Shub [12], if E has degree d then E is conjugate to the mapping $E_d : S^1 \to S^1$ given by $E_d(z) = z^d$ in complex notation. Degree d expanding circle maps are an example of C^{1+} Markov maps with a Markov partition $\{C_0, \ldots, C_{d-1}\}$ of closed intervals such that their end-points are the fixed point $c_0 = C_{d-1} \cap C_0$ of E and its preimages $c_i = C_{i-1} \cap C_i$, for all $0 < i < d$.

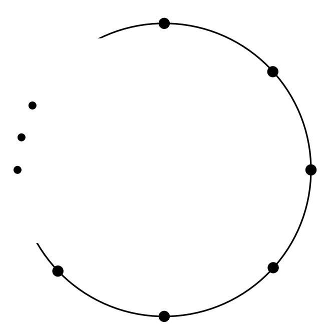

Figure 1: A representation of the circle as a train track made up of cylinders. The larger dots represent the junctions.

We use a (rather trivial) train track T to represent the circle. This consists of d intervals $C_0, \ldots, C_{d-1}$ joined end to end so as to make up a circle (see figure 1). The joining points are called *junctions*. In this case each junction has a very simple structure: it has a smooth *journey* through a junction. The journey defines a smooth chart at the junction. The smooth structure here is just the one coming from the circle. However, in our later examples more complicated structures will be important. The map E respects this smooth structure.

For this example our solenoid function S_E is a mapping from $\{0, \ldots, d-1\}^{\mathbf{Z}_{\geq 0}}$ to the positive reals $\mathbf{R}^+$. It satisfies a simple condition called a *matching condition*. One of our main results is that the correspondence $E \to S_E$ is a bijection from all C^{1+} conjugacy classes of expanding C^{1+} maps of the circle with degree d and all Hölder solenoid functions. These solenoid functions therefore provide the moduli space of smooth conjugacy classes for this problem.

1.1.2 Anosov toral automorphisms.

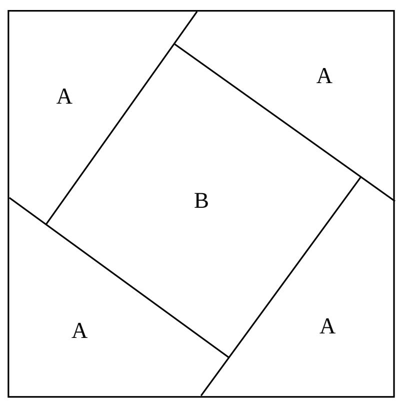

Figure 2: A Markov partition for f into two rectangles A and B. The lines drawn on the torus are segments of the stable and unstable manifolds of the fixed point.

Consider the automorphism f of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ defined by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The unstable (resp. stable) subspace of the origin for f is generated by the vector $v_u = (1, g)$ (resp. $v_s = (-g, 1)$) where $g = (\sqrt{5} - 1)/2$ is the golden mean. They have eigenvalues $\lambda_u = g^{-1}$ and $\lambda_s = -g$. One of the Markov partitions of f is defined in terms of the projection of these subspaces into T^2 as shown in figure 2. The Markov partition of f has two rectangles A and B as shown. The stable and unstable foliations of f are given by the lines parallel to v_s and v_u respectively.

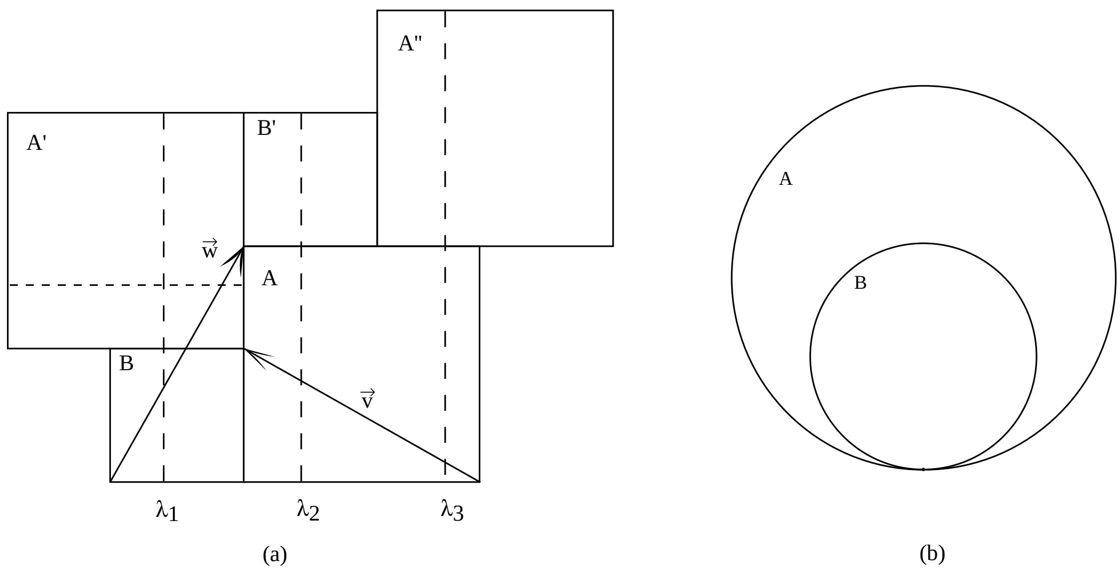

Figure 3: (a) The rectangles A and B of the Markov partition. The vectors $\vec{v}$ and $\vec{w}$ shown generate the lattice which tiles the plane with A and B. The closed one-manifolds λ_1, λ_2 and λ_3 are used as in the text to construct the train track. (b) The train track obtained for the glueing construction using λ_1, λ_2 and λ_3 and the glueings defined by the stable foliation and the translations in the lattice $\mathbf{Z}\vec{v} + \mathbf{Z}\vec{w}$. Note that the only smooth journeys through the junction go from A to A, from A to B and from B to A. There is no smooth journey from B to B.

Alternatively, we can represent this as follows. Let A and B be the squares in the plane shown in figure 3 with areas $a^2 = 1/(1+g^2)$ and $b^2 = g^2/(1+g^2)$ respectively. If we translate these using the vector $\vec{v}$ and $\vec{w}$ shown then they tile the plane. Therefore, $A \bigcup B$ is a fundamental domain for the torus $\mathbf{R}^2/(\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w})$. Consider the linear map L which contract the horizontal by the facter $\lambda_s = -g$ and dilates the vertical by $\lambda_u = g^{-1}$. Under this mapping $\vec{v} \to \vec{w}$ and $\vec{w} \to \vec{v} + \vec{w}$. Therefore, L induces the above automorphism of the torus f and A and B define a Markov partition for L. The stable and unstable foliation are just given by the horizontal and vertical lines.

For each of the stable and unstable foliations we now obtain a train track X and a smooth Markov map $m : X \to X$ on this train track in the following way. Let $A' = A + \vec{w}$, $A'' = A + \vec{v}$ and $B' = B + \vec{v}$ be translation of A and B. Let λ_1, λ_2 and λ_3 be the unstable leaf segments shown in Figure 2. Let $\lambda_{i,\alpha}$ denote the closed segments in λ_i which are in $\alpha \in \{A, A', A'', B, B'\}$. Define maps $g_1 : \lambda_{1,B} \to \lambda_{2,B'}$, $g_2 : \lambda_{1,A'} \to \lambda_{3,A}$, $g_3 : \lambda_{2,A} \to \lambda_{3,A}$ and $g_4 : \lambda_{3,A} \to \lambda_{3,A''}$ by the following prescription. The image of a point x is the unique point in the range which lies in the same stable leaf segment as a translation of x by an element of $\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w}$.

The train track X is the quotient by g_1, g_2, g_3 and g_4 of the disjoint union of λ_1, λ_2 and λ_3. It has the geometry shown in figure 3(b).

The smooth structures of the 1-manifolds λ_1, λ_2 and λ_3 define the smooth structure on X. A path through X is C^r $(r > 1)$ if the induced paths in λ_1, λ_2 and λ_3 are.

Moreover, X is natural for the problems we wish to consider because it has the property given below. Under the projection $\mathbf{R}^2 \to \mathbf{T}^2 = \mathbf{R}^2/(\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w})$ the vertical lines are sent onto the unstable manifolds. We therefore will call these the *unstable leaves*. Let l be the torus lift to $\mathbf{R}^2$ of an unstable leaf in the torus. Define the mapping $\pi_l : l \to X$ by using identification by the translations in $\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w}$ and by identification of points in the same unstable segments contained in A or B. Suppose that we are given an infinite path $j : \mathbf{R} \to X$. Then there is a unique unstable leaf in the torus whose lift l to $\mathbf{R}^2$ is such that the mapping $\pi_l : l \to X$ transverses the two loops of X in the same sequence as the path j. The property of naturalness is the following: There is a C^r diffeomorphism $p : \mathbf{R} \to l$ such that $\pi_l \circ p = j$.

With this smooth structure on X, the map f induces a map $m = m^u : X \to X$ defined as follows. By our construction the map f can be though of a self map of the disjoint union of A and B. Moreover, we have a natural map $\pi : A \bigsqcup B \to X$. The map

m is the unique map which makes the following diagram commute.

$$\begin{array}{ccc} A \bigsqcup B & \xrightarrow{f} & A \bigsqcup B \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{m} & X \end{array}$$

A representation of this map as a map of the interval is shown in figure 4.

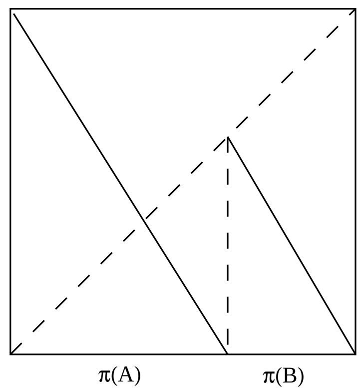

Figure 4: A representation of the Markov map $m : X \to X$ as a map of the interval. The map sends the circle $\tilde{B}$ corresponding to B onto the circle $\tilde{A}$ corresponding to A and maps $\tilde{A}$ onto both $\tilde{A}$ and $\tilde{B}$.

The map m is smooth in the smooth structure of X. It sends smooth paths to smooth paths.

By interchanging stable and unstable (s and u) we can similarly define $m^s : X^s \to X^s$.

This process can be similarly carried out for any C^{1+} Anosov map g which is topologically conjugate to f and defines Markov maps m_g^u and m_g^s. All of the maps m_g^u (resp. m_g^s) are topologically conjugate i.e. if g_1 and g_2 are two such maps then there are canonical homeomorphisms $h_{g_1,g_2}^{u,s} : X_{g_1}^{u,s} \to X_{g_2}^{u,s}$ which conjugates the respective Markov maps. The Markov maps are C^{1+} conjugate if the corresponding homeomorphism is C^{1+}.

The Markov maps that arise from such Anosov systems are easily characterised in terms of the eigenvalues of their fixed points and hence in terms of their solenoid functions. They are also characterised by the fact that the end-points of their cylinders make up an orbit of a C^{1+} homeomorphism of the circle which is conjugate to a rotation. The two diffeomorphisms of the circle obtained in this way correspond to the holonomy maps of the stable and unstable foliation. Looked at from another point of view, this consruction gives precisely those C^{1+} diffeomorphisms of the circle which converge under renormalisation. All of these facts are explained in [5]

In a later paper [4] we will show that there is a one-to-one correspondence between the C^{1+} conjugacy classes of such Anosov maps and pairs of C^{1+} conjugacy classes of such Markov maps. In this paper we characterise the space of C^{1+} conjugacy classes of a wide generalisation of such Markov maps in terms of our Hölder solenoid functions. Thus our results provide a moduli space of pairs of solonoid functions for smooth conjugacy classes of Anosov diffeomorphisms. A clssification in terms of cohomology classes of Hölder cocycles has been proved by Cawley in [1]

Let Σ be the subset of $\ldots \varepsilon_2 \varepsilon_1 \in \{a, b, p_1, p_2, p_3\}^{\mathbf{N}}$ such that for all j, $\varepsilon_{j+1}\varepsilon_j$ is one of ab, ba, bb, bp_1, p_1p_3, p_3p_2 and p_2p_2. Let $\mathcal{S} = \Sigma \times \mathbf{N}$. Then we prove that, in this case, the C^{1+} conjugacy classes of the unstable Markov maps are in one-to-one correspondence with the space of all Hölder functions $s : \mathcal{S} \to \mathbf{R}^+$ which satisfy a certain matching condition and a simple condition corresponding to the abovementioned fixed point eigenvalue condition. A similar result holds for the stable Markov map.

1.1.3 Pronged singularities in pseudo-Anosov maps.

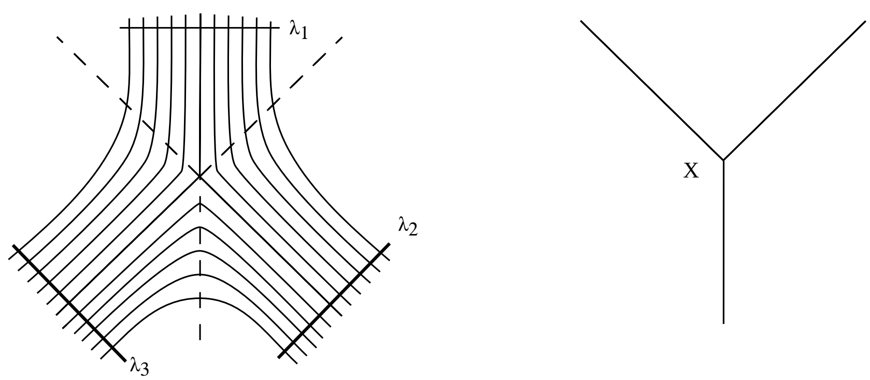

Figure 5: (a) The leaves of the unstable foliation of a pseudo-Anosov diffeomorphism near a 3-pronged singularity. The submanifolds λ_1, λ_2 and λ_3 are the transversals used to construct the train track. (b) The train track X constructed in this way.

Near a 3-pronged singularity the unstable leaves of a pseudo-Anosov map look as in figure 5(a). If we carry out the collapsing procedure analogous to that employed in the last example we obtain a Y-shaped space X as in figure 5(b). Let λ_1, λ_2 and λ_3 be three transversals as shown in 5(a). The manifold structure of these define charts on X by identification of points on the same unstable manifold. But it is clear from figure 5 that these must satisfy the compatibility condition that they agree on the intersection of their domains. To handle the compatibility condition we will introduce the notion of *turntables.* Each junction in our train track will contain a stack of turntables. The charts in each turntable satisfy these strong compatibility conditions.

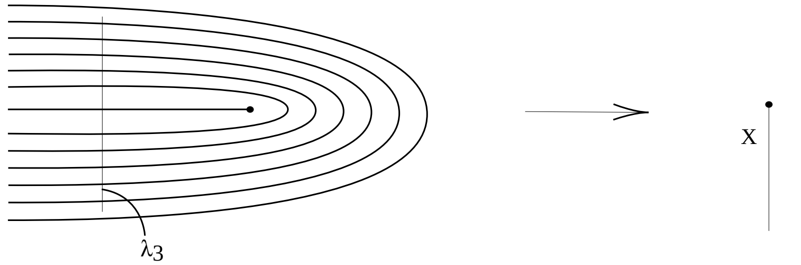

Figure 6: (a) The leaves of the unstable foliation of a pseudo-Anosov diffeomorphism near a 1-pronged singularity. The submanifold λ_1 is the transversal used to construct the train track. (b) The train track X constructed in this way.

For a 1-pronged singularity we obtain the analogous structures shown in figure 6. The unstable manifolds define a map g from λ_1 to itself and the train track X is naturally identified with the quotient λ_1/g. Some more discussion of these two examples is given in subsection 2.1.

1.1.4 Cookie-cutters.

Suppose that I_0 and I_1 are two disjoint closed subintervals of the interval I containing the end-points of $I = [-1, 1]$. A cookie-cutter is a C^{1+} map $F : X \to X$ such that $|dF| > \lambda > 1$ and $F(I_0) = F(I_1) = I$. If

$$\Lambda_n = \{x \in I : F^{j-1}(x) \in I_0 \cup I_1, 1 \leq j \leq n\}$$

then Λ_n consists of 2^n disjoint closed *n-cylinders*

$$I_{\varepsilon_1 \ldots \varepsilon_n} = \{x \in I : F^{j-1}(x) \in I_{\varepsilon_j}, 1 \leq j \leq n\}.$$

Each cylinder

$$I_{\varepsilon_1 \ldots \varepsilon_{n-2}} = I_{\varepsilon_1 \ldots \varepsilon_{n-1}} \cup G_{\varepsilon_1 \ldots \varepsilon_{n-2}} \cup I_{\varepsilon_1 \ldots \varepsilon'_{n-1}}$$

where $G_{\varepsilon_1 \ldots \varepsilon_{n-2}}$ is a *n-gap*. The invariant Cantor set C of F

$$C = \cap_{n \geq 1} \Lambda_n = \{x \in I : F^j x \in I_0 \cup I_1, \text{ for all } j \geq 0\}$$

is constructed inductively by deleting the n-gaps. The smoothness and the expanding property of F implies that the Cantor set C has bounded geometry.

We can regard this as a Markov map on a (rather trivial) train track X as follows. X is the disjoint union of the three closed intervals I_0, $G = I \setminus (\bar{I_0} \cup I_1)$ and I_1 quotient by the junctions $J_1 = \{-1\}$, $J_2 = I_0 \cap G$, $J_3 = G \cap I_1$ and $J_4 = \{1\}$. At the junctions J_2 and J_3, we can define the smooth structure by *journeys*. However, in this case, it is just the smooth structure induced by the inclusion of I_0v G and I_1 into I.

The symbolic space Σ is given by $\{0,1\}^{\mathbf{N}}$, the set of infinite right-handed words $\varepsilon_1\varepsilon_2\ldots$ of 0s and 1s. We add a positive or negative sign to 0 and 1 corresponding to the sign of the derivative of the Markov map F in I_0 and I_1, respectively. There are four possible orderings on the symbolic set Σ corresponding to the two different choices of orientation of the cookie-cutter on each of the two intervals I_0 and I_1.

The mapping $h : \Sigma \to \mathbf{R}$ defined by

$$h(\varepsilon_1\varepsilon_2\ldots) = \bigcap_{n\geq 1} I_{\varepsilon_1\ldots\varepsilon_n}$$

gives an embedding of Σ into $\mathbf{R}$. Moreover, the map h is a topological conjugacy between the shift $\phi : \Sigma \to \Sigma$ and the cookie-cutter $F : \Lambda \to \Lambda$ defined on its invariant set.

We use a train track X to represent the interval I as follows. Let the train track X be the disjoint union of the closed intervals I_0, $G = I \setminus (I_0 \cup I_1)$ and I_1 quotient by the junctions $J_1 = \{-1\}$, $J_2 = I_0 \cap G$, $J_3 = G \cap I_1$ and $J_4 = \{1\}$. At the junctions J_2 and J_3, we define the smooth structure by *journeys*. Each journey is just the identity map from any subset of X containing J_2 or J_3 to I.

For this example our solenoid function S_F is any Hölder continuous mapping from $\{0,1\}^{\mathbf{Z}_{\geq 0}}$ to the positive reals $\mathbf{R}^+$. The fact that it does not have to satisfy a condition as in equation (1) is an advantage when compared to the scaling function. There is a bijection from all C^{1+} conjugacy classes of C^{1+} cookie-cutters and all Hölder solenoid functions.

2 Train tracks.

The underlying space of a traintrack X is a quotient space defined as follows. Consider a finite set of *lines* $l_1, \ldots, l_m$. Each line is path connected one-dimensional closed manifold. The end points $l_i^{\pm}$ of the line l_i are the *termini* of l_i. A *regular point* is a point in X that is not a terminus. The termini are partitioned into junctions J_α. Then X is the quotient space obtained from the disjoint union of the lines by identifying termini in the same junction.

A journey j is a mapping of an interval $I = (t_0, t_n)$ into X with the following properties.

- there is a finite set of times $t_0 < t_1 < \ldots < t_n$ such that $j(t)$ is in a junction if and only if $t = t_i$ for some $0 < i < n$;
- j is a local homeomorphism at all regular points;

- for all small $\varepsilon > 0$ and all $0 < i < n$ the map j gives a local homeomorphism of $(t_i - \varepsilon, t_i]$ and $[t_i, t_i + \varepsilon)$ into the lines containing $j(t_i - \varepsilon)$ and $j(t_i + \varepsilon)$.

Suppose that x and y are termini. If there is a journey $j : I \to X$ such that $j(t_i)$ is in the junction and for some $\varepsilon > 0$ $j(t)$ is in I_x (resp. I_y) when $t \in (t_i - \varepsilon, t_i)$ (resp. $(t_i, t_i + \varepsilon)$) then we write $x \rightleftharpoons y$ and say that there is a *connection* from x to y. If $x \rightleftharpoons x$ then we sat that x is *reversible*. An example of a reversible terminus is given by the train track obtained from a 1-pronged singularity in a pseudo-Anosov diffeomorphism (see figure 6).

Given journeys j_1 and j_2 such that $j_1(s') = j_2(t')$, let $s(t)$ be the unique function defined on a neighbourhood of t' such that $s(t') = s'$ and $j_1(s(t)) = j_2(t)$. Call $s(t)$ the *timetable conversion* of (j_1, j_2) at x. The journeys j_1 and j_2 are C^r *compatible* if the timetable conversion is C^r for all common points x.

A C^r structure on X is defined by giving a compatible set of journeys which pass through every point of X and through every connection. However, it has to satisfy some extra conditions that we now specify.

As explained in subsection 1.1.3 we often require extra constrains at junctions. These are described by *turntables*. Associated to every junction is a set (possibly empty) of turntables. Each turntable τ is a subset of the junction such that if $x, y \in \tau$ then there is a connection between x and y. We adopt the convention that every subset of a turntable is a turntable. A *maximal turntable* is one that is not contained in any bigger one. The *degree* of a turntable is the number of termini in it where each reversible terminus is counted twice.

A smooth structure on the train track X must satisfy the following condition at each turntable τ. If j_1 (resp. j_2) is a C^r journey through the connection from x to y (resp. x to z) and $j_1 = j_2$ on the line terminating in x then $-j_1$ and j_2 define a C^r journey through the connection from y to z. The journey $-j_1$ is the journey j_1 with time reversed.

Definition 1 A C^r *smooth structure* on a train track X is defined by a set of journeys $\{j_\alpha\}$ such that

- every point of X is visited by some journey j_α;
- the journeys $\{j_\alpha\}$ are C^r compatible; and
- the above turntable condition holds.

When we speak of a smooth metric on X, we just mean a metric on the disjoint union of the lines which is a smooth metric on each line.

2.1 Train track obtained by glueing.

A standard way of constructing a train track is given by the following construction.

We are given a finite number of path connected closed one manifolds $\lambda_1, \ldots, \lambda_s$ and a set $\mathcal{D}$ of C^r diffeomorphisms whose domain and ranges are each a closed submanifold of the λ_j. The domain and range can be in the same λ_j.

From the disjoint union of the λ_j we form the quotient space X obtained by identifying all pairs of points which are the form $x, g(x)$ where $g(x) \in \mathcal{D}$. The smooth structure is that defined by the set of projections $\pi_j : \lambda_j \to X$.

Figure 7 gives same examples of local traintrack structures obtained in this way.

Let us use the construction to get a better understanding of the examples given in section 1.1 involving pronged singularities of pseudo-Anosov maps.. For the 3-pronged case, one can see the smooth structure on the Y-shaped space from this figure. We use the glueing construction of subsection 1.1.3. The unstable leafs define the glueing maps $g_{i,j} : \lambda_i \to \lambda_j$ by holonomy. It is clear from this that the smooth structure on the Y is defined by the three charts given by the projection of each λ_i into Y. But from the picture one can see that any two of these determines the third. This is the turntable condition for the Y.

For the single prong or cusp singularity of a pseudo-Anosov map take λ_1 as shown in Figure 6 then there is a single glueing map $g : \lambda_1 \to \lambda_1$ given by the holonomy on leaves. This is shown in figure 7(e).

3 Markov maps.

We now define the notion of a smooth Markov map $F : X \to X$ on the train track X. For such a map the set of lines is partitioned into the subset of *cylinders* C and and the subset of *gaps* G. We let X_0 denote the subspace of X corresponding to the cylinders. The map M does not have to be defined on the gaps. We insist that the lines terminating in a turntable of degree $d > 2$ are all cylinders.

We say that a mapping $F : X \to X$ is *faithful on journeys* at the turntable τ if every short journey through τ is sent to a journey through the image turntable τ' and the preimage of every short journey through τ' is a journey through τ.

A map $F : X \to X$ is *Markov* if

- F is a local homeomorphism on the interior of each cylinder;
- F maps termini to termini;

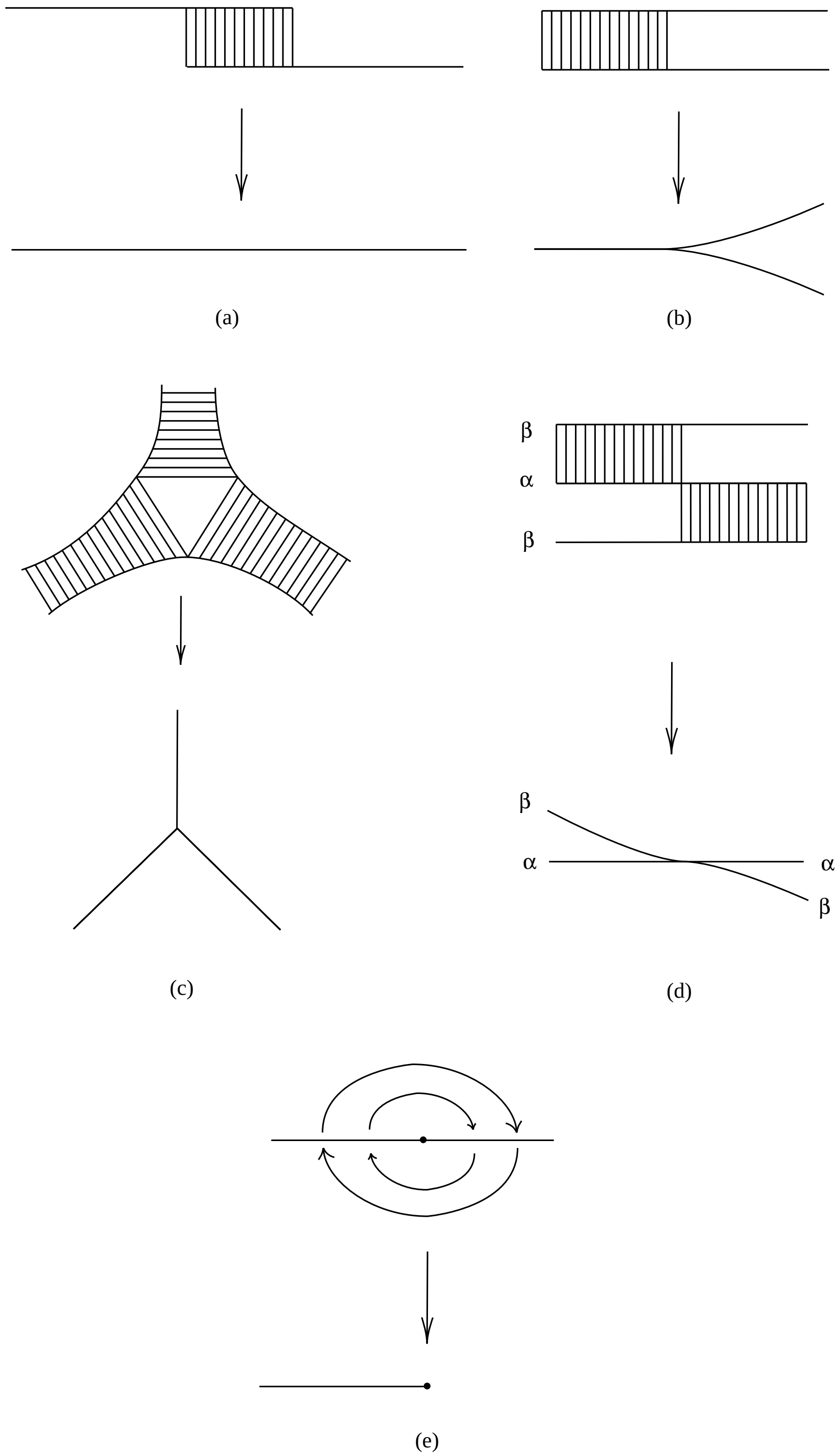

Figure 7: Some local traintracks obtained by glueing. Note that those shown in (c) and (d) have no embedding into Euclidean space.

- F permutes the turntables of degree $d > 2$ and is faithful on journeys at each of them;
- if τ is a maximal turntable of degree 2 then τ is the image of either a regular point or a maximal turntable which has degree 2; and
- for all lines B there exists a cylinder C such that the image of C contains B.

Note that these conditions imply that the image of a cylinder contains all the cylinders that it meets. If F maps the turntable τ to the turntable τ' then we say that F is a C^r diffeomorphism at τ if it maps every C^r journey through τ' C^r diffeomorphically onto a C^r journey through τ'.

If $r > 1$, a C^r Markov map $F : X \to X$ is a Markov map such that

- at every regular point F is a local C^r diffeomorphism;
- F is a C^r diffeomorphism in each turntable of degree $d > 2$;
- if τ is a maximal turntable of degree 2 then τ is the C^r diffeomorphic image of either a regular point or a maximal turntable τ' of degree 2; and
- there exists $\lambda > 1$ and a smooth metric on X such that at every regular point x, $||dF(x)|| > \lambda$.

Let us suppose that the cylinders are indexed by a set S. Thus we denote them by C_a, $a \in S$.

A point $x \in \bigcup_{a \in S} C_a$ is *captured* if for all $m > 0$, $F^m(x) \in \bigcup_{a \in S} C_a$. The set of all captured points in $\bigcup_{a \in S} C_a$ is denoted by $\Lambda = \Lambda_F$. The set of intervals $\{C_a\}$ is called the *Markov partition* of F.

The Markov maps F and G are *topologically conjugate* if there exist a homeomorphism $h : \Lambda_F \to \Lambda_G$ such that $G \circ h = h \circ F$ on Λ_F. If the map h has a C^r extension to X then we say that the map h is a C^r *conjugacy*.

If h ia a mapping of a closed subset X of $\mathbf{R}^n$ inti $\mathbf{R}^m$ then we say that it is C^r if it has a C^r extension to some neighnourhood of X. Moreover, we say that amap h is C^{1+} if it is $C^{1+\varepsilon}$ smooth, for some $0 < \varepsilon < 1$.

Theorem 1 Two C^r Markov maps F and G on a C^r train track are C^r conjugate if and only if they are in the same C^{1+} conjugacy class.

Remark. The optimal result in this direction says that in this theorem C^{1+} can be replaced by uniformly asymptotically affine (uaa). This is proved in [3].

Proof of theorem 1. This theorem is proved by using a blow-down blow-up technique as in the case where X is a one-manifold (e.g. see theorem 27 in [9]). ∎

Symbolic Dynamics.

Given a Markov map F, let $\Sigma_s = \Sigma_s^F$ denote the symbolic set of infinite right-handed words $\underline{\varepsilon} = \varepsilon_1\varepsilon_2\ldots$ such that (i) for all $m \geq 1$, $\varepsilon_m \in S$ and (ii) there exists $x_{\underline{\varepsilon}} \in C$ with the property that $F^m(x_{\underline{\varepsilon}}) \in C_{\varepsilon_m}$ for all $m \geq 1$. We call these words *admissible*.

Endow Σ_s with the usual topology. Let $\Sigma = \Sigma_F$ be the space obtained from Σ_s by identifying $\underline{\varepsilon}$ with $\underline{\varepsilon}'$ if $x_{\underline{\varepsilon}}$ is equal to $x_{\underline{\varepsilon}'}$. If $x_{\underline{\varepsilon}}$ and $x'_{\underline{\varepsilon}}$ are in the same junction of X then $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ are defined to be in the same junction of Σ. If $\{x_{\underline{\varepsilon}_1}, \ldots, x_{\underline{\varepsilon}_n}\}$ is a turntable for X then $\{\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_n\}$ is defined to be a turntable for Σ.

Define the shift $\Phi = \Phi_F : \Sigma \to \Sigma$ by $\Phi(\varepsilon_1\varepsilon_2\ldots) = \varepsilon_2\varepsilon_3\ldots$. As it is well-known, the Markov map F on Λ_F is topologically conjugate to Φ_F on Σ_F.

Two Markov maps can give rise to the same shift map even though they are not topologically conjugate because Σ does not take in account the order of the points in each set $C_a \subset X$. Therefore, we order the points on Σ using the ordering on the corresponding points in the sets $C_a \subset X$. The *symbolic dynamical system* is the set Σ with the shift $\Phi : \Sigma \to \Sigma$.

Lemma 1 The correspondence $F \to \Phi_F$ induces a one-to-one correspondence between topological conjugacy classes of Markov maps and symbolic dynamical systems.

The proof of lemma 1 is simple.

Cylinder structures.

For all $\underline{\varepsilon} \in \Sigma_F$ let $t = \varepsilon_1 \ldots \varepsilon_l$ and define the *l-cylinder* $C_t = C_t^F$ as the closed interval consisting of all $x \in C$ such that for all $1 \leq j \leq l$, $F^j(x) \in C_{\varepsilon_j}$. Suppose that C_t and C_s are two l-cylinders such that (i) C_t and C_s are contained in the same 1-cylinder C_a; (ii) there is no other l-cylinder between C_t and C_s in C_a; (iii) in the interior of the cylinder C_a, $C_t \cap C_s = \emptyset$. Then we define the *l-gap* $C_g = C_{g_{t,s}}$ to be the closed interval between them. An *l-line* is defined to be a l-cylinder or a l-gap. This defines the *cylinder structure* of F.

We say that a cylinder structure has *bounded geometry* if there are constants $c > 0$ and $m > 0$ such that

(i) for all $l > 1$ if D is a l-line and E is the $(l-1)$-cylinder which contains D then $|D|/|E| > c$ and

(ii) if F is the $(l-m)$-cylinder which contains D then $D \neq F$.

4 The scaling function.

Before proceeding to our treatment of the smooth classification problem in terms of the solenoid function, we consider the generalisation of the scaling function defined by Sullivan [13] for cookie-cutters. A C^{1+} Markov map F defines a Hölder scaling function σ_F which is a complete invariant of the C^{1+} conjugacy class of C^{1+} Markov maps (see theorem 2 and theorem 3). We give two simple examples of scaling functions $\sigma_F = \sigma$ which do not have a corresponding C^{1+} Markov map.

For the special case of C^{1+} Markov maps which do not have connections, we prove the one-to-one correspondence between Hölder scaling functions and C^{1+} conjugacy classes of C^{1+} Markov maps without connections. This result is stated in theorem 4.

The metric set $\overline{\Omega}$.

Let us consider the cylinder structure generated by a C^{1+} Markov map F. Define the set $\Omega_n = \Omega_n^F$ as the set of all symbols t corresponding to the n-cylinders and n-gaps.

We also must keep a record of other basic topological information as follows: we record (i) the topological order of all n-cylinders within each 1-cylinder; (ii) which end points of each n-cylinders are junctions and which junction they are.

Define the set $\overline{\Omega} = \overline{\Omega}^F$ as the set of all infinite left-handed words $\overline{t} = \ldots t_n \ldots t_1$ such that for all $n \geq 1$, $t_n \in \Omega_n$ and $F(C_{t_{n+1}}) = C_{t_n}$.

Define the metric $d : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}^+$ as follows. Choose $0 < \mu < 1$. For all $\overline{t}, \overline{s} \in \overline{\Omega}$, the distance $d(\overline{t}, \overline{s})$ is equal to μ^n if and only if $t_n = s_n$ and $t_{n+1} \neq s_{n+1}$.

For all $t \in \Omega_{n+1}$ *the mother* of t is the symbol $m(t) \in \Omega_n$ which has the property that $C_t \subset C_{m(t)}$. For all $\overline{t} \in \overline{\Omega}$ define the *set of children* $\mathcal{C}_{\overline{t}}$ of $\overline{t}$ as the set of all infinite left symbols $\overline{s}$ such that t_n is the mother of s_{n+1} for all $n \geq 1$:

$$\mathcal{C}_{\overline{t}} = \{\overline{s} : m(s_{n+1}) = t_n, \quad \text{for all} \quad n \geq 1\}.$$

Let $\sigma_F : \overline{\Omega} \to \mathbf{R}$ be the well-defined function

$$\sigma_F(\overline{t}) = \lim_{n \to \infty} |C_{t_n}|/|C_{m(t_n)}|.$$

Definition 2 A *scaling function* is a function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ such that for all $\overline{t} \in \overline{\Omega}$

$$\sum_{\overline{s} \in \mathcal{C}_{\overline{t}}} \sigma(\overline{s}) = 1. \tag{1}$$

We say that the scaling function is *Hölder* if it is Hölder continuous in the above metric d.

The equality (1) tells us that there is a corresponding topological Markov map. Since $\overline{\Omega}$ is compact, the scaling function is bounded from zero if and only if, the cylinder structure has bounded geometry. The Hölder continuity of the scaling function corresponds to the C^{1+} smoothness of the Markov map.

Theorem 2 If F is a C^{1+} Markov map then the function $\sigma_F : \overline{\Omega}_F \to \mathbf{R}^+$ is a Hölder scaling function.

Theorem 3 Let F and G be two C^r Markov maps in the same topological conjugacy class. Then F and G are C^r conjugate, where $r > 1$ if and only if the scaling function σ_F is equal to σ_G.

By theorem 3, the Hölder scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ is a complete invariant of the C^{1+} conjugacy classes of C^{1+} Markov maps.

Theorem 4 Given a Hölder scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ with domain $\overline{\Omega}$ corresponding to a topological Markov map without connections then there is a C^{1+} Markov map F with scaling function $\sigma_F = \sigma$.

Corollary 1 There is a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} Markov maps without connections and Hölder scaling functions $\sigma : \overline{\Omega} \to \mathbf{R}^+$ with domain $\overline{\Omega}$ corresponding to topological Markov maps without connections. Moreover, if F and G are C^r Markov maps in the same C^{1+} conjugacy class of C^{1+} Markov maps then they are C^r conjugate.

Proof of corollary 1. This is by theorems 1, 2, 3 and 4. ∎

4.1 A Hölder scaling function without a corresponding smooth Markov map.

In this section we describe one of the problems associated with the scaling function. The general problem is the following. For a given class of smooth Markov maps the scaling function is a complete invariant but it is difficult to determine which scaling function actually arise.

To illustrate this we consider the following simple class of Makov maps of the interval $I = [0,1]$. We consider expanding maps $f : I \to I$ such that for some $0 < r < 1$, $f_0 = f|_{[0,r]}$ (resp. $f_1 = f|_{[r,1]}$) is a C^{1+} diffeomorphism of $[0,r]$ (resp. $[r,1]$) onto I. Clearly, each such map f determines a Hölder scaling function $s_f : \overline{\Omega} = \{0,1\}^{\mathbf{Z}_{>0}} \to \mathbf{R}$. Moreover, theorem 4 asserts that every such function occurs in this way.

However, we now consider the subclass of such mappings which correspond to degree two expanding mappings of the circle. By the above comments the scaling functions for these sre a complete invariant of C^{1+} conjugacy. Moreover, we have the following fact.

Lemma 2 There is a Hölder scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ of the above form such that no C^{1+} expanding map of the circle has σ as its scaling function.

Proof of lemma 2. If F defines a C^{1+} expanding map of the circle then $\sigma_F(\ldots 00) = \sigma_F(\ldots 11)$. But clearly there are Hölder scaling functions in the above class which do not satisfy this property. ∎

Thus if we wish to find a moduli space for expanding maps of the circle, we are naturally lead to the question of which scaling functions occur. It is now easy to see that this problem is not naturally posed in terms of the scaling function. In particular, it is clear that the condition for a scaling function to correspond to a smooth map of the circle contains infinitely many equalities of the form of that used in the proof of the lemma. The natural object with which to consider such problems is the solenoid function.

5 The moduli space of solenoid functions.

Given a C^{1+} Markov map F we construct a pseudo-Hölder solenoid function $\mathrm{s}_F : \mathcal{S}_F \to \mathbf{R}^+$. By construction, there is a one-to-one correspondence between the domains of the solenoid functions and the topological conjugacy classes of Markov maps.

By theorem 6, the pseudo-Hölder solenoid function is a complete invariant of the C^{1+} classes of C^{1+} Markov maps. More importantly, given a pseudo-Hölder solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ we construct a corresponding C^{1+} Markov map F such that $\mathrm{s}_F = \mathrm{s} : \mathcal{S} \to \mathbf{R}^+$. Therefore, the space of the pseudo-Hölder solenoid functions is a *moduli space* for the C^{1+} conjugacy classes of C^{1+} Markov maps.

Moreover, the solenoid function has the following additional advantage over the scaling function. The solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ corresponding to a Markov map without connections does not have to satisfy any constrain such as equality (1).

We use our examples of section 1.1 to give some easy and elucidative examples of solenoid functions and we exemplify the importance of the definition of connections.

In [8], we generalise this concept to give a moduli space for the C^{1+} conjugacy classes of C^{1+} Markov families (see [11]).

The results of this section naturally extend from the C^{1+} smoothness category to the (uaa) uniformly asymptotically affine category replacing the pseudo-Hölder continuity of the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ by the pseudo-continuity of the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ (see [3]).

5.1 The solenoid function.

Firstly, we define the solenoid function $\mathrm{s}_F : \mathcal{A} \to \mathbf{R}^+$ on a set $\mathcal{A}$ which depends on F. We then show how to replace $\mathcal{A}$ by an appropriate symbol space which only depends upon the topological conjugacy class of F.

The function $\mathrm{s}_F : \mathcal{A} \to \mathbf{R}^+$.

Let F be a topological Markov map. A *connection preorbit* $\overline{c}$ of a connection $c_1 = \{c_1^-, c_1^+\} \in C$ is a sequence $\overline{c} = \ldots c_2 c_1$ such that (i) for all $m > 1$, $c_m = \{c_m^-, c_m^+\}$ is a connection or $c_m^- = c_m^+$; (ii) $F(c_m^-) = c_{m-1}^-$ and $F(c_m^+) = c_{m-1}^+$.

Given $n \in \mathbf{N}$ the pair $(\overline{c}, n)$ determines sequences $E^-(\overline{c}, n) = \ldots E_{n+1}^- E_n^-$ and $E^+(\overline{c}, n) = \ldots E_{n+1}^+ E_n^+$ of lines as follows: E_{n+m}^- (resp. E_{n+m}^+) is the $(n+m-1)$-line with c_m^- (resp. c_m^+) as an end-point. We call the pair $(E^-(\overline{c}, n), E^+(\overline{c}, n))$ a *two-line preorbit*.

If c_1 is a preimage connection then the scaling structure of its two-line preorbits $(E^-(\overline{c}, n), E^+(\overline{c}, n))$ with $n > 1$ is determined by those of its image. Therefore, we just need to keep track of the scaling for the two-line preorbits $(E^-(\overline{c}, 1), E^+(\overline{c}, 1))$. On the other hand, if c_1 has no preimage then we must study the scaling of its two-line preorbits $(E^-(\overline{c}, n), E^+(\overline{c}, n))$ for all $n \geq 1$.

Let the set of all *preimage connections* PC be equal to the set of all connections which are C^{1+} preimages of either a connection or a regular point. Therefore, by definition of a Markov map if a connection c is contained in a turntable of degree $d > 2$ then the connection c is a preimage connection. Let the set GC of all *gap connections* be the set of all connections $\{x, y\}$ such that x or y is an extreme point of a gap. Let the set $\mathcal{A} = \mathcal{A}_F$ of F be equal to

$$\mathcal{A} = \{(\overline{c}, n) : \ (c_1, n) \in C \times \{1\} \ \text{ or } \ (c_1, n) \in C \setminus (PC \cup GC) \times \mathbf{N}\}$$

where $\overline{c}$ is a connection preorbit of $c_1 \in C$.

Define the function $\mathrm{s}_F : \mathcal{A} \to \mathbf{R}^+$ by

$$\mathrm{s}_F(\overline{c}, n) = \lim_{m \to \infty} \frac{|E^-_{n+m}|}{|E^+_{n+m}|}.$$

The symbolic definition of the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$.

The solenoid set $\mathcal{S}$ is a symbolic description of the set $\mathcal{A}$. The function $\mathrm{s}_F : \mathcal{A} \to \mathbf{R}^+$ will correspond to the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$.

Let $(E^-(\overline{c}, n), E^+(\overline{c}, n)) = (\ldots E^-_{n+1} E^-_n, \ldots E^+_{n+1} E^+_n)$ be a two-line preorbit.

Let a_m (resp. b_m) be the label of the m-line which contains E^-_{n+m} (resp. E^+_{n+m}). Therefore, $a = a(\overline{c}, n) = \ldots a_2 a_1$ and $b = b(\overline{c}, n) = \ldots b_2 b_1$ are contained in the set $\overline{\Omega}$. The *solenoid set* $\mathcal{S} = \mathcal{S}_F$ is the set

$$\mathcal{S} = \{(a(\overline{c}, n), b(\overline{c}, n), n) : (\overline{c}, n) \in \mathcal{A}\}.$$

For all $(\overline{t}, n) = (\overline{a}, \overline{b}, n) \in \mathcal{S}_F$ adjoin to the symbols a_n and b_n all the order information and all the topological information on the extreme points of the $m+n$-lines $D_{a_m, n} = E^-_{n+m}$ and $D_{b_m, n} = E^+_{n+m}$. This information codes the order of the n-lines in the 1-lines and which lines and points are in which junction.

Lemma 3 The correspondence $F \to \mathcal{S}_F$ induces a one-to-one correspondence between topological conjugacy classes of Markov maps and solenoid sets.

The Proof of lemma 3 follows by construction of the solenoid set $\mathcal{S}$ and by lemma 1.

In section 12 we will introduce a new notation for the solenoid set $\mathcal{S}$ to prove theorems 5, 7, 17, and 18 which is more inteligible for computations.

The turntable condition of a function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$.

Let F be a C^{1+} Markov map. Let $(\overline{c}, n)$ be a two-line preorbit such that the points c^-_m and c^+_m belong to a turntable s_m, for all $m \geq 1$. Since the Markov map F is C^{1+} smooth, the degree of the turntable s_m stabilises to d when $m \geq M$.

Let $e_1^m, \ldots, e_d^m$ be any d connections through the turntables s_m such that the exit $e_j^{+,m}$ of e_j^m is the entrance $e_{j+1}^{-,m}$ of e_{j+1}^m for all $1 \leq j < d$ and the exit $e_d^{+,m}$ of e_d^m is the entrance $e_1^{-,m}$ of e_1^m.

For all $1 \leq j \leq d$, let D_j be a l-line with extreme points $e_j^{-,m}$ and $e_j^{+,m}$. Then the product of the ratios is equal to

$$\prod_{j=1}^{d-1} \frac{|D_{j+1}|}{|D_j|} = \frac{|D_d|}{|D_1|}.$$

This imposes the following turntable condition on the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$. Let $\overline{\varepsilon}_j = \ldots e_j^2 e_j^1$ be a connection preorbit , for all $1 \leq j \leq d$. The *turntable condition* is equal to $\prod_{j=1}^d s(\overline{\varepsilon}_j, n) = 1$.

The matching condition of a function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$.

The ratio between two cylinders D_1 and D_2 at level n is determined by the ratios of all cylinders contained in the union $D_1 \cup D_2$. This imposes the following matching condition on the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$.

For all $(\overline{t}, n) = (\overline{a}, \overline{b}, n) \in \mathcal{S}$, define

$$\mathcal{C}_{(\overline{t},n)'} = \{\overline{z} \in \overline{\Omega} : D_{z_{m+n+1}} \subset D_{a_m,n} \text{ for all } m \geq 1\}$$

and

$$\mathcal{C}_{(\overline{t},n)''} = \{\overline{z} \in \overline{\Omega} : D_{z_{m+n+1}} \subset D_{b_m,n} \text{ for all } m \geq 1\}.$$

For all $\overline{u}, \overline{v} \in \mathcal{C}_{(\overline{t},n)'} \cup \mathcal{C}_{(\overline{t},n)''}$ define

$$\mathrm{s}(\overline{u}, \overline{v}) = \lim_{m \to \infty} \frac{|D_{v_{m+n+1}}|}{|D_{u_{m+n+1}}|}.$$

The function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ satisfies the *matching condition* if and only if for all $(\overline{t}, n) \in \mathcal{S}$ and for all $\overline{u} \in \mathcal{C}_{(\overline{t},n)'} \cup \mathcal{C}_{(\overline{t},n)''}$

$$\frac{\sum_{\overline{v} \in \mathcal{C}_{(\overline{t},n)'}} \mathrm{s}(\overline{u}, \overline{v})}{\sum_{\overline{v} \in \mathcal{C}_{(\overline{t},n)''}} \mathrm{s}(\overline{u}, \overline{v})} = \mathrm{s}(\overline{t}, n).$$

Definition 3 A function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is a *solenoid function* if and only if satisfies the matching and the turntable conditions.

A pseudo-Hölder continuous function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$.

We define a metric d on the solenoid set $\mathcal{S}$ as follows. The distance between $(\overline{s}, n)$ and $(\overline{z}, n)$ in $\mathcal{S}$ is equal to μ^{m+n}, if and only if $s_m = z_m$ and $s_{m+1} \neq z_{m+1}$, where $0 < \mu < 1$. Otherwise, the distance between two elements of $\mathcal{S}$ is equal to infinity.

The solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is *pseudo-Hölder continuous* if and only if there is a constant $c > 0$ and $0 < \alpha < 1$ such that for all $(\overline{s}, n), (\overline{z}, n) \in \mathcal{S}$

$$\left|1 - \frac{\mathrm{s}(\overline{s}, n)}{\mathrm{s}(\overline{z}, n)}\right| < c(d((\overline{s}, n), (\overline{z}, n)))^{\alpha}.$$

Theorem 5 The C^{1+} Markov map F defines the pseudo-Hölder solenoid function $\mathrm{s} = \mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ by

$$\mathrm{s}(\overline{a}, \overline{b}, n) = \lim_{m \to \infty} \frac{|D_{a_m,n}|}{|D_{b_m,n}|}.$$

Theorem 6 The correspondence $F \to \mathrm{s}_F$ induces a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} Markov maps and pseudo-Hölder solenoid functions. Moreover, if F and G are C^r Markov maps in the same C^{1+} conjugacy class of Markov maps then they are C^r conjugate.

By compactness of the subset $\mathcal{S}_{GC}$ of all $(\overline{z}, 1) \in \mathcal{S}$, the solenoid function restricted to $\mathcal{S}_{GC}$ is bounded from zero and infinity which corresponds to the bounded geometry of the cylinder structure of the Markov map. The matching condition of the solenoid function corresponds to the existence of a topological Markov map. The turntable condition correspond to the existence of journeys through the turntables. The pseudo-Hölder property of the solenoid function corresponds to the existence of a C^{1+} Markov map.

If the solenoid function is bounded from zero then a pseudo-Hölder solenoid function is Hölder continuous. Otherwise, the solenoid function is just pseudo-Hölder continuous, see example 5.2.

If the set of all preimage connections PC is equal to the set of all connections for a C^{1+} Markov map F then the corresponding solenoid function is Hölder continuous. That is the case of cookie-cutters, tent maps on train tracks, expanding circle maps, horocycle maps and Markov maps generated by pseudo-Anosov maps, for example.

5.2 Examples of solenoid functions for Markov maps.

We will give some examples of solenoid sets and solenoid functions for Markov maps. The examples we give of solenoid functions are very simple, usually they have a much richer structure. For example, the scaling functions related to renormalisable structures usually have a lot of self-similarities (see pinto:thesis).

Cookie-cutters.

Suppose that C_0 and C_1 are two disjoint closed subintervals of the interval C containing the end-points of $C = [-1, 1]$. Let the train track X be the disjoint union of the closed intervals I_0, $G = I \setminus (I_0 \cup I_1)$ and I_1 with junctions $J_1 = \{-1\}$, $J_2 = I_0 \cap G$, $J_3 = G \cap I_1$ and $J_4 = \{1\}$. The set of all connections is equal to $\{J_2, J_3\}$. Let $F : X \to X$ be a cookie-cutter. Let $S_F = \{0, 1\}$ and G be the 1-gap between C_0 and C_1. Add the symbol 0 to the gap point $g_0 = C_0 \cap G$ and associate the symbol 1 to the gap point $g_1 = C_1 \cap G$. Associate the information that the Markov branches F_0 and F_1 are orientation preserving or orientation reversing to the symbols 0 and 1. Let the symbol sequence $\ldots \varepsilon_2 \varepsilon_1 \in \{0,1\}^{\mathbf{N}}$ represent the image of the gap point g_{ε_1} by the inverse Markov branches $F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_2}^{-1}$ for all $m > 1$. Then the solenoid set $\mathcal{S}$ is represented by the set

$$\mathcal{S} = \{0,1\}^{\mathbf{N}}.$$

Let F be the following cookie-cutter.

$$F(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 3x - 2 & x \in [\frac{2}{3}, 1] \end{cases}$$

The solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is defined as follows. For all $\ldots \varepsilon_2 \varepsilon_1 \in \mathcal{S}$,

$$\mathrm{s}(\ldots \varepsilon_2 0) = \frac{|F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_2}^{-1}(C_0)|}{|F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_2}^{-1}(G)|} = 3$$

and

$$\mathrm{s}(\ldots \varepsilon_2 1) = \frac{|F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_2}^{-1}(C_1)|}{|F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_2}^{-1}(G)|} = 2.$$

Tent maps defined on an interval.

Let X be the train track containing the cylinders $C_0 = [a_0, b_0]$ and $C_1 = [a_1, b_1]$ with the junctions $p_2 = \{a_0, a_0\}$, $p_3 = \{b_0, a_1\}$ and $p_4 = \{b_1, b_1\}$. Let the set of all connections be equal to $\{p_3\}$. Let $T : X \to X$ be the C^{1+} *tent map* such that (i) $dT > \lambda > 1$ in C_0 and $dT < \lambda < -1$ in C_1; (ii) $T(C_0) = F(C_1) = I$. Let the set S_T be equal to $\{0, 1\}$. Therefore, the set of all preimage connections PC is equal to the set of all connections. Associate the information that the Markov branches F_0 is orientation preserving to the symbol 0 and that F_1 is orientation reversing to the symbol 1. Let the symbol sequences

$$(\ldots \varepsilon_1, n) \in \{0,1\}^{\mathbf{N}} \times \mathbf{N}$$

represent the image of the two n-cylinders with connection p_3 by the inverse Markov branches $T_{\varepsilon_m}^{-1} \circ \ldots \circ T_{\varepsilon_1}^{-1}$ for all $m > 1$. Then the solenoid set $\mathcal{S}$ is represented by the set

$$\mathcal{S} = \{0,1\}^{\mathbf{N}} \times \mathbf{N}.$$

Let T be the following example of a tent map.

$$T(x) = \begin{cases} 3x & x \in [0, \frac{1}{3}] \\ -\frac{3}{2}x + \frac{3}{2} & x \in [\frac{1}{3}, 1] \end{cases}$$

For this example the solenoid function $s_T : \mathcal{S} \to \mathbf{R}^+$ is the constant function $s_T = 2$.

A smooth Markov map with a pseudo-Hölder solenoid function which is not Hölder continuous.

Let X be the train track containing the cylinders $C_0 = [a_0, b_0]$ and $C_1 = [a_1, b_1]$ with the junctions $p_2 = \{a_0, a_0\}$, $p_3 = \{b_0, a_1\}$ and $p_4 = \{b_1, b_1\}$. Let the set of all connections be equal to $\{p_3\}$. Let $F : X \to X$ be the C^{1+} Markov map such that $dF > \lambda > 1$ in $C_0 \cup C_1$ and $F(C_0) = F(C_1) = I$. Therefore, the set of all preimage connections PC is equal to the set of all connections. Let $S_F = \{0, 1\}$ and let the symbol sequences

$$(\ldots \varepsilon_1, n) \in \{0, 1\}^{\mathbf{N}} \times \mathbf{N}$$

represent the image of the two n-cylinders with connection p_3 by the inverse Markov branches $F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_1}^{-1}$ for all $m > 1$. Then the solenoid set $\mathcal{S}$ is represented by the set

$$\mathcal{S} = \{0, 1\}^{\mathbf{N}} \times \mathbf{N}.$$

Let F be the following example.

$$F(x) = \begin{cases} 3x & x \in [0, \frac{1}{3}] \\ \frac{3}{2}x - \frac{1}{2} & x \in [\frac{1}{3}, 1] \end{cases}$$

For all $(\ldots \varepsilon_1, n) \in \mathcal{S}$ the solenoid function $s_F(\ldots \varepsilon_1, n) = 2^n$. Therefore, the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is pseudo-Hölder continuous but it is not Hölder continuous.

Tent maps defined on a train track.

Let X be the train track containing the cylinders $C_0 = [a_0, b_0]$ and $C_1 = [a_1, b_1]$ with the junctions $p_2 = \{a_0, a_0\}$, $p_3 = \{b_0, a_1\}$ and $p_4 = \{b_1, b_1\}$. Let the set of all connections be equal to the set of all junctions. Let $T : X \to X$ be a tent map. Therefore, the set of all preimage connections PC is equal to the set of all connections. Let the sequence $\ldots \varepsilon_2 \varepsilon_1 3$ correspond to the preimage

$$F_{\varepsilon_m}^{-1} \ldots F_{\varepsilon_2}^{-1} F_{\varepsilon_1}^{-1}(p_3)$$

of p_3, for all $m \geq 1$. Let the sequence $\ldots \varepsilon_2 \varepsilon_1 34$ correspond to the preimage

$$F_{\varepsilon_m}^{-1} \ldots F_{\varepsilon_2}^{-1} F_{\varepsilon_1}^{-1} F^{-1}(p_4)$$

of p_4, for all $m \geq 1$. Let the sequence $\ldots\varepsilon_2\varepsilon_1 342\ldots 2$ correspond to the preimage

$$F_{\varepsilon_m}^{-1}\ldots F_{\varepsilon_2}^{-1}F_{\varepsilon_1}^{-1}F^{-1}F_1^{-1}F_0^{-1}\ldots F_0^{-1}(p_2)$$

of p_2, for all $m \geq 1$. The solenoid set $\mathcal{S}$ can be represented as the subset of all sequences

$$\ldots\varepsilon_2\varepsilon_1 \in \mathcal{S} \subset \{0,1,2,3,4\}^{\mathbf{N}} \times \mathbf{N}$$

such that 0 or 1 is followed by 0 or 1 or 3; 3 is followed by 4; 4 is followed by 2; 2 is followed by 2.

Let T be the affine tent map

$$T(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 2-2x & x \in [\frac{1}{2}, 1] \end{cases} .$$

For this example the solenoid function $s_T : \mathcal{S} \to \mathbf{R}^+$ is the constant function $s_T = 1$.

Expanding maps of the circle.

Let $E : S^1 \to S^1$ be a C^{1+} expanding map of the circle with degree d. Let $S = \{0,\ldots,d-1\}$ and for all $0 < i < d$ let c_i be the connection $c_i = C_{i-1} \cap C_i$ and c_0 be the connection $c_0 = C_{d-1} \cap C_0$. The set of all connections is equal to the set of all preimage connections.

Let the symbol sequence

$$\ldots\varepsilon_1\varepsilon_0 \in \{0,\ldots,d-1\}^{\mathbf{N}}$$

represent the image of the connection c_{ε_0} by the inverse Markov branches $F_{\varepsilon_m}^{-1} \circ \ldots \circ F_{\varepsilon_1}^{-1}(c_{\varepsilon_0})$ for all $m > 1$. Then the solenoid set $\mathcal{S}$ is represented by the set

$$\mathcal{S} = \{0,\ldots,d-1\}^{\mathbf{N}}.$$

Let E be the expanding map of the circle with degree d,

$$E(x) = dx(\text{mod } 1).$$

For this example the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is the constant function $s_F = 1$.

5.3 The horocycle maps and the diffeomorphisms of the circle.

Let $H : X \to X$ be the horocycle map such that (i) $H_a(C_a) = C_b$ and the extreme points a_1 and a_2 of C_a are send into $H(a_1) = b_2$ and $H(a_2) = b_1$; (ii) $H_b(C_b) = X$ and the extreme points b_1 and b_2 of C_a are send into $H(b_1) = a_2$ and $H(b_2) = b_1$. The set of all connections $p_1 = \{a_1, b_2\}$, $p_2 = \{a_2, b_1\}$ and $p_3 = \{b_1, b_2\}$ is equal to the set of all preimage connections.

Let the sequence $\ldots \varepsilon_2 \varepsilon_1 b p_1$ correspond to the preimage

$$H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1}(p_1)$$

of p_3, for all $m \geq 1$. Let the sequence $\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3$ correspond to the preimage

$$H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1} H^{-1}(p_3)$$

of p_3, for all $m \geq 1$. Let the sequence $\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3 p_2 \ldots p_2$ correspond to the preimage

$$H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_b^{-1} H^{-1} \ldots H^{-1}(p_2)$$

of p_2, for all $m \geq 1$. The solenoid set $\mathcal{S}$ can be represented as the subset of all sequences

$$\ldots \varepsilon_2 \varepsilon_1 \in \mathcal{S} \subset \{a, b, p_1, p_3, p_2\}^{\mathbf{N}} \times \mathbf{N}$$

such that a is followed by b; b is followed by a or b or p_1; p_1 is followed by p_3; p_3 is followed by p_2; p_2 is followed by p_2.

Let H be the horocycle map corresponding to the rigid golden rotation R_g, where g is the golden number

$$H(x) = \begin{cases} -gx + 1 & x \in C_b = [0, \frac{1}{g}] \\ -gx + g & x \in C_a = [\frac{1}{g}, 1]. \end{cases}$$

For this example the solenoid function $\mathrm{s}_H : \mathcal{S} \to \mathbf{R}^+$ is the following quasi-constant function. For all $\ldots \varepsilon_2 \varepsilon_1 b p_1 \in \mathcal{S}$

$$\mathrm{s}_H(\ldots \varepsilon_2 \varepsilon_1 b p_1) = \frac{|H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1}(C_a)|}{|H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1}(C_b)|} = \frac{1}{g}.$$

For all $\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3 \in \mathcal{S}$

$$\mathrm{s}_H(\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3) = \frac{|H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1} H^{-1}(C_b)|}{H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_{\varepsilon_1}^{-1} H_b^{-1} H^{-1}(C_b)|} = 1.$$

For all $\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3 p_2 \ldots p_2 \in \mathcal{S}$

$$\mathrm{s}_H(\ldots \varepsilon_2 \varepsilon_1 b p_1 p_3 p_2 \ldots p_2) = \frac{|H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_b^{-1} H^{-1} \ldots H^{-1}(C_a)|}{|H_{\varepsilon_m}^{-1} \ldots H_{\varepsilon_2}^{-1} H_b^{-1} H^{-1} \ldots H^{-1}(C_b)|} = \frac{1}{g}.$$

The horocycle map $H : X \to X$ generates a C^{1+} diffeomorphism $f : S^1 \to S^1$ of the circle with golden rotation number (see [4]). The generalisation of the horocycle map H to horocycle families $(H_m)_{m \in \mathbf{Z}}$ generates C^{1+} diffeomorphisms of the circle f with bounded rotation number (see [4] and [10]).

Using theorem 6, we prove in [4] that the C^{1+} Anosov maps are in one-to-one correspondence with pairs of horocycle solenoid functions (see also [1]for a classification in terms of cohomology classes of Hölder cocycles). Using the holonomy map we prove a rigidity result for Anosov maps.

5.4 The importance of the connections of a smooth Markov map.

The connection $c = C_a \cap C_b$ between two cylinders C_a and C_b expresses the existence of a smooth structure on a neighbourhood of $c = C_a \cap C_b$ in $C_a \cup C_b$. We give an example which illustrates the importance of the connection property.

Let $F : I \to I$ be the C^{1+} Markov map defined by

$$F = \begin{cases} -\frac{3}{2}x + 3 & x \in [0,2] \\ 2x - 1 & x \in [2,3] \\ \frac{3}{2}x - \frac{9}{2} & x \in [3,5] \end{cases}$$

with Markov partition $C_0 = [0,2]$, $C_1 = [2,3]$ and $C_2 = [3,5]$. The set of connections is $C_F = \{2\}$ and the set of preimage connections PC_F is equal to the empty set.

Define the homeomorphism $h : [0,5] \to J$ such that (i) h is equal to a smooth map h_1 in the set $[0,3]$ and to a smooth map h_2 in the set $[3,5]$; (ii) h is not smooth at the point 3.

Let $G = h \circ F \circ h^{-1} : J \to J$ be the smooth Markov map with Markov partition $B_0 = [h(0), h(2)]$, $B_1 = [h(2), h(3)]$ and $B_2 = [h(3), h(5)]$. The set of connections is $C_G = \{h(2)\}$ and the set of preimage connections PC_G is equal to the empty set.

Since the point 3 is not a connection of F and the point $h(3)$ is not a connection of G then the map h is a smooth conjugacy between the smooth Markov map F and the smooth Markov map G, even if h is not smooth at the point 3 in the usual sense.

The scaling function $\sigma_F : \overline{\Omega}_F \to \mathbf{R}^+$ is equal to $\sigma_G : \overline{\Omega}_G \to \mathbf{R}^+$ and the solenoid function $\mathrm{s}_F : \mathcal{S}_F \to \mathbf{R}^+$ is equal to $\mathrm{s}_G : \mathcal{S}_G \to \mathbf{R}^+$.

6 The α-solenoid functions.

We now turn from questions concerning C^{1+} smoothness to the corresponding questions for $C^{1+\alpha}$ smoothness for a specific value of α.

By theorem 6, given a solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ there is a C^{1+} Markov map F such that $\mathrm{s}_F = \mathrm{s}$ and vice-versa. Below, we construct a metric $d = d_s$ on the solenoid set $\mathcal{S}$ using the cylinder structure of F.

The solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is an α-solenoid function if and only if the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is β pseudo-Hölder continuous with respect to the metric d, for all $0 < \beta < \alpha$. This property is independent of the C^{1+} Markov map F used to define the metric d, if $\mathrm{s}_F = \mathrm{s}$.

A map F is $C^{1+\alpha^-}$ *smooth* if and only if F is $C^{1+\beta}$ smooth for all $0 < \beta < \alpha \leq 1$. Given a $C^{1+\alpha^-}$ Markov map F the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is an α-solenoid function. Given an α-solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ there is a $C^{1+\alpha^-}$ Markov map F such that $\mathrm{s}_F = \mathrm{s}$. If the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is not an η-solenoid function for some $\eta > \alpha > 0$ then there is not a $C^{1+\eta}$ Markov map F such that $\mathrm{s}_F = \mathrm{s}$.

The α-solenoid functions are in one-to-one correspondence with the $C^{1+\alpha^-}$ conjugacy classes of $C^{1+\alpha^-}$ Markov maps. Thus, the space of the α-solenoid functions is a moduli space for the $C^{1+\alpha^-}$ conjugacy classes of $C^{1+\alpha^-}$ Markov maps.

The Lipschitz metric d_L.

Let F be a topological Markov map. For all $(\overline{t}, n) = (\overline{a}, \overline{b}, n)$ and $(\overline{s}, n) \in \mathcal{S}$, the distance $d_L((\overline{t}, n), (\overline{s}, n))$ is equal to

$$d_L((\overline{t}, n), (\overline{s}, n)) = |D_{a_m,n}| + |D_{b_m,n}|$$

if and only if $t_m = s_m$ and $t_{m+1} \neq s_{m+1}$. Otherwise, the distance between two elements of $\mathcal{S}$ is infinity.

By theorem 6, given a solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ there is a C^{1+} Markov map F such that $\mathrm{s}_F = \mathrm{s}$. The solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is β pseudo-Hölder continuous if and only if there is a constant $c_\beta > 0$ such that for all $(\overline{t}, n), (\overline{s}, n) \in \mathcal{S}$

$$\left| 1 - \frac{\mathrm{s}(\overline{t}, n)}{\mathrm{s}(\overline{s}, n)} \right| < c_\beta (d_L((\overline{t}, n), (\overline{s}, n)))^\beta .$$

Definition 4 An *α-solenoid function* $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is a solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ which is β pseudo-Hölder continuous for all $0 < \beta \leq \alpha \leq 1$.

Lemma 4 The definition of the α-solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is independent of the C^{1+} Markov map F used to define the metric d_L, if $\mathrm{s}_F = \mathrm{s}$.

Theorem 7 Given a $C^{1+\alpha^-}$ Markov map F the solenoid function $\mathrm{s} = \mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is β pseudo-Hölder continuous for all $0 < \beta < \alpha$.

Theorem 8 The correspondence $F \to \mathrm{s}_F$ induces a one-to-one correspondence between $C^{1+\alpha^-}$ smooth conjugacy classes of $C^{1+\alpha^-}$ Markov maps and α-solenoid functions.

Corollary 2 If the Markov map F defines an α-solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ which is not $\eta > \alpha$ pseudo-Hölder continuous then the Markov map F is not $C^{1+\eta}$ smooth.

7 The solenoid function and α-determination.

Let F be a topological Markov map such that $s_F : \mathcal{S} \to \mathbf{R}^+$ is a pseudo-Hölder continuous solenoid function. Below, we define the α-determination property of the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$. It has the following properties: (i) if the Markov map F is $C^{1+\alpha^-}$ smooth then the solenoid function s_F has the α-determination property; (ii) if the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is α-determined then there is a $C^{1+\alpha^-}$ Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$. Trivially, if the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is α-determined then the solenoid function s_F is an α-solenoid function.

Similarly, we define the determination property of the solenoid function such that (i) if the Markov map F is C^{1+} smooth then the solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is determined; (ii) if the solenoid function s_F is determined then there is a C^{1+} Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

α-determination.

Let us consider the cylinder structure generated by a topological Markov map F. Define the *pre-solenoid function* $s_F : \mathcal{S} \times \mathbf{N} \to \mathbf{R}^+$ by

$$s_F(\overline{a}, \overline{b}, n, p) = \frac{|D_{a_p,n}|}{|D_{b_p,n}|}.$$

Definition 5 Let F be a topological Markov map such that $s_F : \mathcal{S} \to \mathbf{R}^+$ is a solenoid function. The solenoid function s_F is α-*determined* (resp. α-*strong;y determined*) if and only if for all $0 < \beta < \alpha$ (resp. $0 < \beta \le \alpha$) there are constants $c, c_\beta > 0$ such that for all $(\overline{t}, n) = (\overline{a}, \overline{b}, n) \in \mathcal{S}$

$$\left| 1 - \frac{s_F(\overline{t}, n)}{s_F(\overline{t}, n, p)} \right| < c_\beta (|D_{a_p,n}| + |D_{b_p,n}|)^\beta .$$

Theorem 9 If the Markov map F is $C^{1+\alpha^-}$ smooth then the solenoid function s_F is α-determined.

Theorem 10 If the pseudo-Hölder solenoid function s_F is α-determined then there is a $C^{1+\alpha^-}$ Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

By theorem 8 and theorem 9, if the solenoid function s_F is α-determined then it is an α-solenoid function s_F.

Corollary 3 If the pseudo-Hölder solenoid function s_F is α-strongly determined then there is a $C^{1+\alpha}$ Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

Definition 6 Let F be a topological Markov map such that s_F is a solenoid function. The solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is *determined* if and only if there is a constant $c > 0$ and $0 < \lambda < 1$ such that for all $(\overline{t}, n) = (\overline{a}, \overline{b}, n) \in \mathcal{S}$

$$\left| 1 - \frac{\mathrm{s}_F(\overline{t}, n)}{\mathrm{s}_F(\overline{t}, n, p)} \right| < c\lambda^{n+p}.$$

Theorem 11 If the Markov map F is C^{1+} smooth then the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ is determined.

Theorem 12 If the pseudo-Hölder solenoid function s_F is determined then there is a C^{1+} smooth Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

8 Smoothness of Markov maps and geometry of the cylinder structures.

The relevance of this section is to built the bridge between this paper and [7]. The theorems of this section are used several times in the proofs of the theorems of this paper. On the other hand, the proofs of the theorems of this section are done in [7].

In this section, we give a balanced equivalence between the smoothness of the Markov map F and the geometry of the respective cylinder structure. We give a balanced equivalence between the geometry of the cylinder structures corresponding to the Markov maps F and G and the smoothness of the conjugacy h between the Markov maps F and G.

8.1 The solenoid property of a cylinder structure.

Definition 7 The cylinder structure generated by the Markov map F has the *α-solenoid property* (resp. *α-strong solenoid property*) if and only if for all $0 < \beta < \alpha$ (resp. $0 < \beta \leq \alpha$) there are constants $c, c_\beta > 0$ such that for all $(\overline{t}, n, p) = (\overline{a}, \overline{b}, n, p) \in \mathcal{S} \times \mathbf{N}$, (i) $\mathrm{s}_F(\overline{t}, 1, p) > c$; (ii)

$$\left| 1 - \frac{\mathrm{s}_F(\overline{t}, n, p)}{\mathrm{s}_F(\overline{t}, n, p+1)} \right| \leq c_\beta(|D_{a_p,n}| + |D_{b_p,n}|)^\beta.$$

Theorem 13 A $C^{1+\alpha^-}$ Markov map F generates a cylinder structure with the α-solenoid property. A cylinder structure with the α-solenoid property generates a $C^{1+\alpha^-}$ Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

Corollary 4 A cylinder structure with the α-strong solenoid property generates a $C^{1+\alpha}$ Markov map G such that $\Lambda_G = \Lambda_F$ and $G|\Lambda_G = F|\Lambda_F$.

Definition 8 The cylinder structure generated by the Markov map F has the *solenoid property* if and only if there are constants $c_1, c_2 > 0$ and $0 < \lambda < 1$ such that for all $(\bar{t}, n, p) \in \mathcal{S} \times \mathbf{N}$, (i) $\mathrm{s}_F(\bar{t}, 1, p) > c_1$; (ii)

$$\left|1 - \frac{\mathrm{s}_F(\bar{t}, n, p)}{\mathrm{s}_F(\bar{t}, n, p+1)}\right| \leq c_2 \lambda^{n+p}.$$

Theorem 14 A C^{1+} Markov map F generates a cylinder structure with the solenoid property and vice-versa.

8.2 The solenoid equivalence between cylinder structures.

Let F and G be two topologically conjugate Markov maps. Therefore, the sets $\mathcal{S}_F \times \mathbf{N}$ and $\mathcal{S}_G \times \mathbf{N}$ are equal.

Definition 9 The cylinder structures generated by the C^{1+} Markov maps F and G are *solenoid equivalent* if and only if there are constants $c_1, c_2 > 0$ and $0 < \lambda < 1$ such that for all $(\bar{t}, n, p) \in \mathcal{S} \times \mathbf{N}$, (i) $\mathrm{s}_F(\bar{t}, 1, p) > c_1$; (ii)

$$\left|1 - \frac{\mathrm{s}_F(\bar{t}, n, p)}{\mathrm{s}_G(\bar{t}, n, p)}\right| \leq c_2 \lambda^{n+p}.$$

Theorem 15 Let F and G be C^r Markov maps in the same topological conjugacy class. The conjugacy between F and G is C^r smooth if and only if the cylinder structures generated by F and G are solenoid equivalent.

9 The canonical set C of charts.

By lemma 3, the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ defines a symbolic set Σ_F corresponding to a topological Markov map F. We construct a set $\mathbf{C}$ of canonical charts of the symbolic set Σ_F such that (i) for all $x \in \Sigma_F$ and for the shift $F(x) = \Phi(x) \in \Sigma_F$ there are charts $c : \Sigma_c \to \mathbf{R}^+$ and $e : \Sigma_e \to \mathbf{R}^+$ in a neighbourhood of x and in a neighbourhood of $F(x)$, respectively, such that the Markov map F is affine with respect to the charts c and e; (ii) the solenoid function s_F is equal to the solenoid function s; (iii) the composition map $d \circ c^{-1}$ between any two charts c and d is a smooth map, whenever defined.

We define the canonical charts $c : \Sigma_c \to \mathbf{R}^+$ by the respective pre-solenoid functions $\mathrm{s}_c : \underline{\Omega}_c \to \mathbf{R}^+$ up to affine transformations as follows.

Let $c = (\bar{t}, 1) = (\bar{a}, \bar{b}, 1) \in \mathcal{S}$. The *pre-solenoid set* $\underline{\Omega}_c = \cup_{n \geq 1} \underline{\Omega}_{c,n}$ is the set of all points $(\bar{s}, m, p) = (\bar{v}, \bar{z}, m, p) \in \mathcal{S} \times \mathbf{N}$ such that the cylinders $D_{v_{p+j-1}, m}, D_{z_{p+j-1}, m} \subset D_{a_j, 1} \cup D_{b_j, 1}$ for all $j \geq 1$. Let $\underline{\Omega}_{c,n} \subset \underline{\Omega}_c$ be the set of all symbols $(\bar{s}, m, p) \in \underline{\Omega}_c$

such that $m+j+p=n$. The *pre-solenoid function* $\mathrm{s}_c : \underline{\Omega}_c \to \mathbf{R}^+$ is defined by $\mathrm{s}_c(\overline{s}, m, p) = \mathrm{s}(\overline{s}, m)$.

The domain Σ_c of the canonical chart c is the set of all symbols $\varepsilon_1 \varepsilon_2 \ldots \in \Sigma_F$ such that ε_1 is equal to a_1 or b_1.

The matching condition implies that the pre-solenoid functions $\mathrm{s}_c : \underline{\Omega}_c \to \mathbf{R}^+$ define a cylinder structure, i.e. the set of all extreme points of the cylinders at level n is contained in the set of all extreme points of the cylinders at level $n+1$, for all $n \geq 1$. The turntable condition implies the existence of turntable journeys in the neighbourhood of each turntable.

Theorem 16 Given a solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$, the composition map $c_d \circ c_c^{-1}$ between two canonical charts c and d is a C^{1+} smooth map, whenever defined. If $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is an α-solenoid function then the composition map $c_d \circ c_c^{-1}$ is a $C^{1+\alpha^-}$ smooth map whenever defined.

Theorem 17 Given a solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ there is a C^{1+} Markov map F such that $\mathrm{s}_F = \mathrm{s} : \mathcal{S} \to \mathbf{R}^+$.

Theorem 18 Given an α-solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ there is a $C^{1+\alpha^-}$ Markov map F such that $\mathrm{s}_F = \mathrm{s} : \mathcal{S}^F \to \mathbf{R}^+$.

10 Proof of theorems 2, 3 and 4.

First, we introduce the following notation that we will use in the proof of the theorems.

The pair $(t_n, s_n) \in \Omega_n \times \Omega_n$ is an *adjacent symbol* if and only if the lines C_{t_n} and C_{s_n} have a common point or one of the extreme points x of C_{t_n} and one of the extreme points y of C_{s_n} are a connection $\{x, y\}$. The set $\underline{\Omega}_n \subset \Omega_n \times \Omega_n$ is the set of all adjacent symbols.

If f and g are functions of a variable x with domain Δ, then we write $\mathcal{O}_x(f(x)) = \mathcal{O}_x(g(x))$ with constant d if

$$d^{-1} < \frac{|f(x)|}{|g(x)|} < d$$

for all $x \in \Delta$. Often we will drop the reference to d.

If it is obvious which variable x is involved then we use the notation $\mathcal{O}(f(x))$ instead. Thus if a_n and b_n are sequences then $\mathcal{O}(a_n) = \mathcal{O}(b_n)$ means a_n/b_n and b_n/a_n are bounded away from 0 independently of n. The notation $f(x) = \mathcal{O}(g(x))$ means the same thing as $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$.

Similarly, $f(x) \leq \mathcal{O}(g(x))$ with constant d means $|f(x)/g(x)| < d$ for all $x \in \Delta$.

Proof of theorem 2. We are going to prove that if the Markov map F is C^{1+} smooth then $\sigma_F : \overline{\Omega} \to \mathbf{R}^+$ is a Hölder scaling function.

By theorem 3 in [7] the Markov partition of F has the (1+)-*scaling property*, i.e. there is $0 < \lambda < 1$ such that for all $\overline{t} = \ldots t_1 \in \overline{\Omega}$,

$$\left|1 - \frac{\sigma_F(t_n)}{\sigma_F(t_{n-1})}\right| \leq \mathcal{O}(\lambda^n).$$

Thus, for all $p, q > n > 0$,

$$\frac{\sigma_F(t_p)}{\sigma_F(t_q)} \in 1 \pm \mathcal{O}(\lambda^n).$$

Therefore, the limit $\sigma_F(\overline{t})$ is well defined and

$$\frac{\sigma_F(\overline{t})}{\sigma_F(t_n)} \in 1 \pm \mathcal{O}(\lambda^n). \tag{2}$$

By smoothness of the Markov map F and the expanding nature of F there is $\delta > 0$ such that for all $t \in \Omega = \cup_{n \geq 1} \Omega_n$, $\sigma_F(t) > \delta$. Therefore, for all $\overline{t} \in \overline{\Omega}$

$$\sigma(\overline{t}) = \lim_{n \to \infty} \sigma_F(t_n) > \delta.$$

Let $0 < \varepsilon \leq 1$ be such that $\lambda \leq \mu^\varepsilon$. For all $\overline{t}, \overline{s} \in \overline{\Omega}$ such that $t_n = s_n$ and $t_{n+1} \neq s_{n+1}$, we have by inequality (2)

$$|\sigma_F(\overline{t}) - \sigma_F(\overline{s})| \leq \mathcal{O}(\lambda^n) \leq \mathcal{O}\left((\mu^\varepsilon)^n\right) \leq \mathcal{O}\left(\left(d(\overline{t}, \overline{s})\right)^\varepsilon\right).$$

Therefore, the function σ_F is Hölder continuous in $\overline{\Omega}$.

For all $\overline{t} \in \overline{\Omega}$ the set $\mathcal{C}_{\overline{t}}$ is bounded and

$$\sum_{\overline{s} \in \mathcal{C}_{\overline{t}}} \sigma_F(\overline{s}) = \lim_{n \to \infty} \sum_{\overline{s} \in \mathcal{C}_{\overline{t}}} \sigma_F(s_{n+1}) = 1.$$

Therefore, $\sigma_F : \overline{\Omega} \to \mathbf{R}^+$ is a Hölder scaling function. ∎

Proof of theorem 3. If the Markov maps F and G are C^{1+} conjugate then by theorem 1 they are C^r-conjugate . By theorem 3 in [7] any two C^{1+} Markov maps F and G are C^{1+} conjugate if and only if the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent.

Therefore, we will prove that the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent if and only if the scaling function $\sigma_F : \overline{\Omega}_F \to \mathbf{R}^+$ is equal to the scaling function $\sigma_G : \overline{\Omega}_G \to \mathbf{R}^+$.

We will give the definitions of (1+)-scale equivalence and (1+)-connection equivalence during the proof.

By theorem 3 in [7] and smoothness of the Markov maps F and G their cylinder structures have the (1+)-scale property and the (1+)-connection property. Therefore, they satisfy inequality (2) of the proof of theorem 2.

Let us prove that if the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent then they define the same scaling function.

Since F and G are topologically conjugate then they define the same set $\overline{\Omega} = \overline{\Omega}_F = \overline{\Omega}_G$.

Since the cylinder structures of F and G are (1+)-*scale equivalent* there is $0 < \lambda < 1$ such that for all $\overline{t} = \ldots t_1 \in \overline{\Omega}$

$$\left|1 - \frac{\sigma_F(t_n)}{\sigma_G(t_n)}\right| \leq \mathcal{O}(\lambda^n). \tag{3}$$

By inequalities (2) and (3), for all $\overline{t} = \ldots t_1 \in \overline{\Omega}$ and for all $n > 0$

$$\begin{aligned} \frac{\sigma_F(\overline{t})}{\sigma_G(\overline{t})} &= \frac{\sigma_F(\overline{t})}{\sigma_F(t_n)} \frac{\sigma_F(t_n)}{\sigma_G(t_n)} \frac{\sigma_G(t_n)}{\sigma_G(\overline{t})} \\ &\in (1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n)) \\ &\subset 1 \pm \mathcal{O}(\lambda^n). \end{aligned}$$

On letting n converge to infinity, we obtain that the scaling functions $\sigma_F : \overline{\Omega} \to \mathbf{R}^+$ and $\sigma_G : \overline{\Omega} \to \mathbf{R}^+$ are equal.

Let us prove that if the scaling functions $\sigma_F : \overline{\Omega} \to \mathbf{R}^+$ and $\sigma_G : \overline{\Omega} \to \mathbf{R}^+$ are equal then the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent.

For all $t_n \in \Omega_n$ choose $\overline{t} = \ldots t_n \ldots t_1 \in \overline{\Omega}$. Since $\sigma_F(\overline{t}) = \sigma_G(\overline{t})$ and by inequality (2),

$$\begin{aligned} \frac{\sigma_F(t_n)}{\sigma_G(t_n)} &= \frac{\sigma_F(t_n)}{\sigma_F(\overline{t})} \frac{\sigma_F(\overline{t})}{\sigma_G(\overline{t})} \frac{\sigma_G(\overline{t})}{\sigma_G(t_n)} \\ &\in (1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n)) \\ &\subset 1 \pm \mathcal{O}(\lambda^n). \end{aligned} \tag{4}$$

The cylinder structures of F and G are (1+)-scale equivalent.

For all $t \in \Omega_n$ denote the cylinders C_t^F by C_t and the cylinders C_t^G by D_t. Let us prove that the cylinder structures F and G are (1+)-*connection equivalent*, i.e. for all $(t_n, s_n) \in \underline{\Omega}_n$

$$\frac{|C_{t_n}|}{|C_{s_n}|} \frac{|D_{s_n}|}{|D_{t_n}|} \in 1 \pm \mathcal{O}(\lambda^n).$$

For all adjacent symbols $(t_n, s_n) \in \underline{\Omega}_n$ choose

$$\overline{t} = \ldots t_n \ldots t_1, \overline{s} = \ldots s_n \ldots s_1 \in \overline{\Omega}$$

such that (i) $(t_l, s_l) \in \underline{\Omega}_l$, (ii) there is $0 < k \leq n$ such that $m^k(t_l) = m^k(s_l)$ for all l large enough. Denote $m^i(t_l)$ by t^i and $m^i(s_l)$ by s^i, for all $i = 0, \ldots, k$.

Let H be the Markov map F or G. By the definition of (1+)-*connection property* of the cylinder structure of F and G, there is $0 < \lambda < 1$ such that, for all $l > n$,

$$\left| 1 - \frac{|C^H_{t_n}|}{|C^H_{s_n}|} \frac{|C^H_{s_l}|}{|C^H_{t_l}|} \right| \leq \mathcal{O}(\lambda^n). \tag{5}$$

By inequalities (4) and (5)

$$\begin{aligned} \frac{|C_{t_n}|}{|C_{s_n}|} \frac{|D_{s_n}|}{|D_{t_n}|} &= \frac{|C_{t_n}|}{|C_{s_n}|} \frac{|C_{s_l}|}{|C_{t_l}|} \frac{|C_{t_l}|}{|C_{t^k}|} \frac{|C_{s^k}|}{|C_{s_l}|} \frac{|C_{t^k}|}{|C_{s^k}|} \\ & \quad \frac{|D_{s^k}|}{|D_{t^k}|} \frac{|D_{s_l}|}{|D_{s^k}|} \frac{|D_{t^k}|}{|D_{t_l}|} \frac{|D_{t_l}|}{|D_{s_l}|} \frac{|D_{s_n}|}{|D_{t_n}|} \\ &\in (1 \pm \mathcal{O}(\lambda^n)) \prod_{i=0}^{k} \left(\frac{\sigma_F(t^i)}{\sigma_G(t^i)} \frac{\sigma_G(s^i)}{\sigma_F(s^i)} \right) \\ &\subset (1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^{l-k})). \end{aligned}$$

On letting l tend to infinity, we obtain that the cylinder structures of F and G are (1+)-connection equivalent. ∎

Proof of theorem 4. We are going to prove that given a scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ corresponding to a topological Markov map without connections it corresponds to the scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ a C^{1+} Markov map without connections.

By theorem 3 in [7] given a cylinder structure without connections and with (1+)-scale property and bounded geometry then there is a C^{1+} Markov map F without connections which generates this cylinder structure. We will define (1+)-scale property during the proof.

Define the *pre-scaling function* $\sigma : \Omega \to \mathbf{R}^+$ as follows. For all $n > 1$ and for all $t_{n-1} \in \Omega_{n-1}$ choose $\overline{t} = \ldots t_{n-1} \ldots t_1 \in \overline{\Omega}$. For all $\overline{s} \in \mathcal{C}_{\overline{t}}$ define $\sigma(s_n) = \sigma(\overline{s})$.

Since the scaling function is bounded from zero, trivially the pre-scaling function is bounded from zero. Thus, the cylinder structure corresponding to the pre-scaling function $\sigma : \Omega \to \mathbf{R}^+$ has bounded geometry.

We are going to prove that this cylinder structure has the (1+)-scaling property, i.e. there is $0 < \mu < 1$ such that for all $n > 1$ and for all $s \in \Omega_n$

$$\left| 1 - \frac{\sigma(s)}{\sigma(\phi(s))} \right| \leq \mathcal{O}(\mu^n).$$

For all $n > 1$ and for all $s_n \in \Omega_n$ choose $\overline{s} = \ldots s_n s_{n-1} \ldots s_1 \in \overline{\Omega}$. Since the scaling function is Hölder continuous and it is bounded away from zero

$$\begin{aligned}
\frac{\sigma(s_n)}{\sigma(s_{n-1})} &\in \frac{\sigma(\overline{s}) \pm \mu^n}{\sigma(\overline{s}) \pm \mu^{n-1}} \\
&\subset 1 \pm \mathcal{O}(\mu^n).
\end{aligned}$$

By theorem 3 in [7] there is a C^{1+} Markov map F which generates a cylinder structure with a pre-scaling function equal to $\sigma : \Omega \to \mathbf{R}^+$. By construction of the C^{1+} Markov map F the scaling function $\sigma_F : \overline{\Omega}_F \to \mathbf{R}^+$ of F is equal to the scaling function $\sigma : \overline{\Omega} \to \mathbf{R}^+$ ∎

11 Proof of theorems 13, 14 and 15.

We introduce the following new concepts. The set $\underline{\Omega}_n^s \subset \underline{\Omega}_n$ is the set of all adjacent symbols (t, s) such that the cylinders C_t and C_s have a common regular point or one of the extreme points x of C_t and one of the extreme points y of C_s are a preimage connection $\{x, y\}$. The set $\underline{\Omega}_n^g \subset \underline{\Omega}_n$ is the set of all adjacent symbols (t, s) such that C_t or C_s is a gap and C_t and C_s have a common point.

For all $t, t' \in \Omega_F$ such that $t = t_1, \ldots, t_p = t'$, where $(t_i, t_{i+1}) \in \underline{\Omega}$ for all $1 \le i < p$, define $\mathrm{s}(t, t) = 1$ and

$$\mathrm{s}(t, t') = \prod_{i=1}^{p-1} \mathrm{s}(t_i, t_{i+1}) = \prod_{i=1}^{p-1} \frac{|C_{t_i}|}{|C_{t_{i+1}}|} = \frac{|C_t|}{|C_{t'}|}.$$

For all $t \in \Omega$, the set $\mathcal{B}_t$ of *brothers* of t is the set of all symbols $s \in \Omega$ such that t and s have the same mother. Since the Markov map F has a finite number of branches the cardinality of the set $\mathcal{B}_t$ is bounded independently of the symbol t.

By an easy algebraic manipulation the pre-scaling function $\sigma : \Omega \to \mathbf{R}^+$ is defined at t by

$$(\sigma(t))^{-1} = \sum_{s \in \mathcal{B}_t} \mathrm{s}(s, t).$$

Proof of theorem 13. Let us prove that the Markov map F is $C^{1+\alpha^-}$ smooth if and only if the cylinder structure of F has the α-solenoid property. By theorem 2 in [7] the Markov map F is $C^{1+\alpha^-}$ smooth if and only if the cylinder structure of F has the $(1+\alpha)$-scale property, the $(1+\alpha)$-connection property and has bounded geometry.

It is easy to check that property (i) of the α-solenoid property implies that the cylinder structure of F has bounded geometry and vice-versa. Therefore, we will prove that the cylinder structure of F has the α-solenoid property if and only if the cylinder structure of F has the $(1+\alpha)$-scale property and the $(1+\alpha)$-connection property.

We will give the definitions of $(1+\alpha)$-scale property and $(1+\alpha)$-connection property during the proof.

We now prove that if the cylinder structure of F has the α-solenoid property, then it has the $(1+\alpha)$-scale property and the $(1+\alpha)$-connection property.

Let $\phi(t) \in \Omega$ be such that $F(C_t) = C_{\phi(t)}$. The cylinder structure of F has the $(1+\alpha)$-*scale property* if and only if for all $0 < \beta < \alpha$ and for all $t \in \Omega$

$$\left| \frac{\sigma(t)}{\sigma(\phi(t))} - 1 \right| \leq \mathcal{O}(|C_t|^{\beta}).$$

Since the cylinder structure of F has the α-solenoid property

$$\begin{aligned} \frac{\sigma(t)}{\sigma(\phi(t))} &= \frac{\sum_{s\in\mathcal{B}_t} \mathrm{s}(\phi(s),\phi(t))}{\sum_{s\in\mathcal{B}_t} \mathrm{s}(s,t)} \\ &\in \frac{\sum_{s\in\mathcal{B}_t} \mathrm{s}(s,t)(1 \pm \mathcal{O}(|C_{m(t)}|^{\beta}))}{\sum_{s\in\mathcal{B}_t} \mathrm{s}(s,t)} \\ &\subset 1 \pm \mathcal{O}(|C_{m(t)}|^{\beta}) \\ &\subset 1 \pm \mathcal{O}(|C_t|^{\beta}). \end{aligned}$$

The cylinder structure of F has the $(1+\alpha)$-*connection property* if and only if for all $0 < \beta < \alpha$ and for all $(t,s) \in \underline{\Omega}_n^s$

$$\left| \frac{|C_t|}{|C_s|} \frac{|C_{\phi(s)}|}{|C_{\phi(t)}|} - 1 \right| \leq \mathcal{O}((|C_t| + |C_s|)^{\beta}).$$

By the α-solenoid property the cylinder structure of F has the $(1+\alpha)$-connection property.

Let us prove that if the cylinder structure of F has the $(1+\alpha)$-scale property and the $(1+\alpha)$-connection property then the cylinder structure of F has the α-solenoid property.

By the $(1+\alpha)$-connection property of the cylinder structure of F for all $0 < \beta < \alpha$ and for all $(t,s) \in \underline{\Omega}_n^s$

$$\frac{\mathrm{s}(t,s)}{\mathrm{s}(\phi(t),\phi(s))} \in 1 \pm \mathcal{O}((|C_t| + |C_s|)^{\beta}).$$

By the $(1+\alpha)$-scale property and by bounded geometry for all $0 < \beta < \alpha$ and for all $(t,s) \in \underline{\Omega}_n^g$

$$\begin{aligned} \frac{\mathrm{s}(t,s)}{\mathrm{s}(\phi(t),\phi(s))} &= \frac{\sigma(t)}{\sigma(\phi(t))} \frac{\sigma(\phi(s))}{\sigma(s)} \\ &\in 1 \pm \mathcal{O}(|C_{m(t)}|^{\alpha}) \\ &\subset 1 \pm \mathcal{O}((|C_t| + |C_s|)^{\alpha}). \end{aligned}$$

Therefore, the cylinder structure of F has the α-solenoid property. ∎

Proof of corollary 4. The proof follows in a similar way to the proof of theorem 13 using $\beta = \alpha$ in the definitions of $(1+\alpha)$-scale property and $(1+\alpha)$-connection property and using the corollary of theorem 2 in [7]. ∎

Proof of theorem 14. Let us prove that the Markov map F is C^{1+} smooth if and only if the cylinder structure of F has the solenoid property. By the corollary 1 in [7] the Markov map F is C^{1+} smooth if and only if the cylinder structure of F has the (1+)-scale property, the (1+)-connection property and has bounded geometry.

It is easy to check that property (i) of the solenoid property implies that the cylinder structure of F has bounded geometry and vice-versa.

Therefore, we will prove that the cylinder structure of F has the solenoid property if and only if the cylinder structure of F has the (1+)-scale property and the (1+)-connection property.

We will give the definitions of (1+)-scale property and (1+)-connection property during the proof.

We now prove that if the cylinder structure of F has the solenoid property, then it has the (1+)-scale property and the (1+)-connection property.

The cylinder structure of F has the (1+)-*scale property* if and only if there is $0 < \lambda < 1$ such that for all $t \in \Omega$

$$\left| \frac{\sigma(t)}{\sigma(\phi(t))} - 1 \right| \leq \mathcal{O}(\lambda^n).$$

Since the cylinder structure of F has the solenoid property,

$$\begin{aligned}
\frac{\sigma(t)}{\sigma(\phi(t))} &= \frac{\sum_{s \in \mathcal{B}_t} \mathrm{s}(\phi(s), \phi(t))}{\sum_{s \in \mathcal{B}_t} \mathrm{s}(s,t)} \\
&\in \frac{\sum_{s \in \mathcal{B}_t} \mathrm{s}(s,t)(1 \pm c\lambda^n))}{\sum_{s \in \mathcal{B}_t} \mathrm{s}(s,t)} \\
&\subset 1 \pm \mathcal{O}(\lambda^n).
\end{aligned}$$

The cylinder structure of F has the (1+)-*connection property* if and only if there is $0 < \lambda < 1$ such that for all $(t,s) \in \underline{\Omega}_n^s$

$$\left| \frac{|C_t|}{|C_s|} \frac{|C_{\phi(s)}|}{|C_{\phi(t)}|} - 1 \right| \leq \mathcal{O}(\lambda^n).$$

Since the cylinder structure of F has the solenoid property, trivially it has the (1+)-connection property.

Let us prove that if the cylinder structure of F has the (1+)-scale property and the (1+)-connection property then it has the solenoid property.

By the (1+)-connection property of the cylinder structure of F there is $0 < \lambda < 1$ such that for all $(t, s) \in \underline{\Omega}_n^s$

$$\frac{\mathrm{s}(t,s)}{\mathrm{s}(\phi(t),\phi(s))} \in 1 \pm \mathcal{O}(\lambda^n).$$

By the (1+)-scale property and by bounded geometry for all $(t, s) \in \underline{\Omega}_n^g$

$$\begin{aligned}\frac{\mathrm{s}(t,s)}{\mathrm{s}(\phi(t),\phi(s))} &= \frac{\sigma(t)}{\sigma(\phi(t))}\frac{\sigma(\phi(s))}{\sigma(s)} \\ &\in 1 \pm \mathcal{O}(\lambda^n).\end{aligned}$$

Therefore, the cylinder structure of F has the solenoid property. ∎

Proof of theorem 15. By theorem 3 in [7] the Markov map F and G are C^{1+} conjugate if and only if the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent. Therefore, we will prove that the cylinder structures of F and G are solenoid equivalent if and only if they are (1+)-scale equivalent and (1+)-connection equivalent. We will give the definitions of (1+)-scale equivalence and (1+)-connection equivalence during the proof.

First, we prove that if the cylinder structures of F and G are solenoid equivalent then they are (1+)-scale equivalent and (1+)-connection equivalent.

They are (1+)-*scale equivalent* if and only if there is $0 < \lambda < 1$ such that for all $n > 1$ and for all $t \in \Omega_n$

$$\left|\frac{\sigma_F(t)}{\sigma_G(t)} - 1\right| \le \mathcal{O}(\lambda^n).$$

Since the cylinder structures of F and G are solenoid equivalent

$$\begin{aligned}\frac{\sigma_F(t)}{\sigma_G(t)} &= \frac{\sum_{s\in\mathcal{B}_t}\mathrm{s}_G(s,t)}{\sum_{s\in\mathcal{B}_t}\mathrm{s}_F(s,t)} \\ &\in \frac{\sum_{s\in\mathcal{B}_t}\mathrm{s}_F(s,t)(1\pm\mathcal{O}(\lambda^n))}{\sum_{s\in\mathcal{B}_t}\mathrm{s}_F(s,t)} \\ &\subset 1 \pm \mathcal{O}(\lambda^n).\end{aligned}$$

Now, we prove that the cylinder structures of F and G are (1+)-connection equivalent.

For all $t \in \Omega$ denote the cylinders C_t^F by C_t and the cylinders C_t^G by D_t.

The cylinder structures of F and G are (1+)-*connection equivalent* if and only if there is $0 < \lambda < 1$ such that for all $n > 1$ and for all $(t, s) \in \underline{\Omega}_n^s$

$$\left|\frac{|C_t|}{|C_s|}\frac{|D_s|}{|D_t|} - 1\right| \le \mathcal{O}(\lambda^n).$$

Since the cylinder structures of F and G are solenoid equivalent, they are (1+)-connection equivalent.

Let us prove that if the cylinder structures of F and G are (1+)-scale equivalent and (1+)-connection equivalent then they are solenoid equivalent.

Since they are (1+)-connection equivalent, there is $0 < \lambda < 1$ such that for all $n > 1$ and for all $(t, s) \in \underline{\Omega}_n^s$

$$\frac{\mathrm{s}_F(t,s)}{\mathrm{s}_G(t,s)} \in 1 \pm \mathcal{O}(\lambda^n).$$

By (1+)-scale equivalence for all $(t, s) \in \underline{\Omega}_n^g$

$$\begin{aligned} \frac{\mathrm{s}_F(t,s)}{\mathrm{s}_G(t,s)} &= \frac{\sigma_F(t)}{\sigma_G(t)} \frac{\sigma_G(s)}{\sigma_F(s)} \\ &\in 1 \pm \mathcal{O}(\lambda^n). \end{aligned}$$

Therefore, the cylinder structures of F and G are solenoid equivalent. ∎

12 Proof of theorems 5, 7, 16 and 17.

We are going to state and prove theorems 5, 17, 7 and 18. In the proof of these theorems we will use the following notation for the solenoid set $\mathcal{S} = \mathcal{S}_F$ which is more intelligible for computations. The solenoid set $\mathcal{S} = \mathcal{S}_F$ is the set of all pairs

$$(\overline{t}, \overline{s}) = \ldots t_1, \ldots s_1) \in \overline{\Omega} \times \overline{\Omega}$$

of two-lines preorbits such that there exists $N_{\overline{t},\overline{s}} > 0$ with the property that for all $n \geq N_{\overline{t},\overline{s}}$, $(t_n, s_n) \in \underline{\Omega}_n^s \times \underline{\Omega}_n^g$ if and only if $n > N_{\overline{t},\overline{s}}$. The set $\mathcal{S}_{GC} \subset \mathcal{S}$ is the set of all $(\overline{t}, \overline{s}) \in \mathcal{S}$ such that $N_{\overline{t},\overline{s}} = 1$.

Proof of theorem 5. By theorem 14 and by the smoothness of the Markov map F, the cylinder structure of F has the solenoid property. Thus, there is $0 < \lambda < 1$ such that for all $(\overline{t}, \overline{s}) \in \mathcal{S}$ and for all $n \geq N_{\overline{t},\overline{s}}$,

$$\left| 1 - \frac{\mathrm{s}(t_n, s_n)}{\mathrm{s}(t_{n+1}, s_{n+1})} \right| \leq \mathcal{O}(\lambda^n).$$

Thus, for all $p, q > n \geq N_{\overline{t},\overline{s}} \geq 1$,

$$\frac{\mathrm{s}(t_p, s_p)}{\mathrm{s}(t_q, s_q)} \in 1 \pm \mathcal{O}(\lambda^n).$$

Therefore, the function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is well defined and

$$\frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(t_n, s_n)} \in 1 \pm \mathcal{O}(\lambda^n). \tag{6}$$

Let $0 < \alpha \leq 1$ be such that $\lambda \leq \mu^{\alpha}$. Let $(\overline{t}, \overline{s}), (\overline{t}', \overline{s}') \in \mathcal{S}$ be such that $N_{\overline{t},\overline{s}} = N_{\overline{t}',\overline{s}'}$ and possess the property that $t_n = t'_n$, $s_n = s'_n$, and $t_{n+1} \neq t'_{n+1}$ or $s_{n+1} \neq s'_{n+1}$, for some $n \geq N_{\overline{t},\overline{s}}$. By inequality (6), the function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is pseudo-Hölder continuous

$$\begin{aligned} \left| 1 - \frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(\overline{t}', \overline{s}')} \right| &\leq \left| 1 - \frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(t_n, s_n)} \frac{\mathrm{s}(t'_n, s'_n)}{\mathrm{s}(\overline{t}', \overline{s}')} \right| \\ &\leq \mathcal{O}\left((d((\overline{t}, \overline{s}), (\overline{t}', \overline{s}')))^{\alpha} \right). \end{aligned}$$

By definition of the pre-solenoid function, for all element $(\overline{z}, \overline{s}) \in \mathcal{S}$ such that the elements $(\overline{t}^1, \overline{t}^2) \ldots, (\overline{t}^p, \overline{t}^{p+1}) \in \mathcal{S}$ are in the same turntable and $\overline{z} = \overline{t}^1$ and $\overline{s} = \overline{t}^{p+1}$, the value

$$\prod_{i=1}^{p} \mathrm{s}(\overline{t}^i, \overline{t}^{i+1}) = \lim_{n \to \infty} \prod_{i=1}^{p} \mathrm{s}(t_n^i, t_n^{i+1}) = \lim_{n \to \infty} \mathrm{s}(z_n, s_n) = \mathrm{s}(\overline{z}, \overline{s}).$$

Therefore, the function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ satisfies the turntable condition.

By definition of the pre-solenoid function, for all $(\overline{t}, \overline{s}) \in \mathcal{S}$ and for all $\overline{u} \in \mathcal{C}_{\overline{t}} \cup \mathcal{C}_{\overline{s}}$

$$\begin{aligned} \frac{\sum_{\overline{t}' \in \mathcal{C}_{\overline{t}}} \mathrm{s}(\overline{u}, \overline{t}')}{\sum_{\overline{s}' \in \mathcal{C}_{\overline{s}}} \mathrm{s}(\overline{u}, \overline{s}')} &= \lim_{n \to \infty} \frac{\sum_{\overline{t}' \in \mathcal{C}_{\overline{t}}} \mathrm{s}(u_n, t'_n)}{\sum_{\overline{s}' \in \mathcal{C}_{\overline{s}}} \mathrm{s}(u_n, s'_n)} \\ &= \lim_{n \to \infty} \mathrm{s}(t_n, s_n) \\ &= \mathrm{s}(\overline{t}, \overline{s}). \end{aligned}$$

The function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ satisfies the matching condition. Therefore, the function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is a pseudo-Hölder solenoid function. ∎

Proof of theorem 7. By theorem 13, the cylinder structure of F has the α-solenoid property. Thus, for all $(\overline{t}, \overline{s}) \in \mathcal{S}$ and for all $n \geq N_{\overline{t},\overline{s}}$,

$$\left| 1 - \frac{\mathrm{s}(t_n, s_n)}{\mathrm{s}(t_{n+1}, s_{n+1})} \right| \leq \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^{\beta}).$$

By the expanding property of the Markov map F, for all $p, q > n \geq N_{\overline{t},\overline{s}} \geq 1$,

$$\frac{\mathrm{s}(t_p, s_p)}{\mathrm{s}(t_q, s_q)} \in 1 \pm \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^{\beta}).$$

Therefore,

$$\frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(t_n, s_n)} \in 1 \pm \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^{\beta}). \tag{7}$$

Let $(\overline{t}, \overline{s}), (\overline{u}, \overline{v}) \in \mathcal{S}$ be such that $N_{\overline{t},\overline{s}} = N_{\overline{u},\overline{v}}$ and have the property that $t_n = u_n$, $s_n = v_n$, and $t_{n+1} \neq u_{n+1}$ or $s_{n+1} \neq v_{n+1}$, for some $n \geq N_{\overline{t},\overline{s}} = N_{\overline{u},\overline{v}}$. By inequality (7)

$$\begin{aligned} \left| 1 - \frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(\overline{u}, \overline{v})} \right| &\leq \left| 1 - \frac{\mathrm{s}(\overline{t}, \overline{s})}{\mathrm{s}(t_n, s_n)} \frac{\mathrm{s}(u_n, v_n)}{\mathrm{s}(\overline{u}, \overline{v})} \right| \\ &\leq \mathcal{O}\left((d((\overline{t}, \overline{s}), (\overline{u}, \overline{v})))^{\beta} \right). \end{aligned}$$

Therefore, the solenoid function $s : \mathcal{S} \to \mathbf{R}^+$ is an α-solenoid function. ∎

Proof of theorem 16. By theorem 3 in [7] the composition map $c \circ d^{-1}$ is a smooth map if and only if the cylinder structures of the canonical charts c and d are (1+)-scale equivalent and (1+)-connection equivalent. Similarly to the proof of theorem 15, the (1+)-scale equivalence and (1+)-connection equivalence are equivalent to the following solenoid equivalence defined by equation (8).

For all $n \geq 1$ and for all $(w_n, v_n) \in \underline{\Omega}_{c,n} \cap \underline{\Omega}_{d,n}$ let $(\overline{w} = \ldots w_n \ldots w_1, \overline{v} = \ldots v_n \ldots v_1) \in \mathcal{S}$. By the pseudo-Hölder continuity of the solenoid function, there is $0 < \lambda < 1$ and there are constants $c_1, c_2 > 0$

$$\begin{aligned} \frac{s_c(w_n, v_n)}{s_d(w_n, v_n)} &\in \frac{s(\overline{w}, \overline{v})(1 \pm c_1 \lambda^n)}{s(\overline{w}, \overline{v})(1 \pm c_1 \lambda^n)} \\ &\subset 1 \pm c_1 \lambda^n. \end{aligned} \tag{8}$$

and $s_c(w_n, v_n) > c_2$. Therefore, the cylinder structures of the canonical charts c and d are solenoid equivalent.

By theorem 3 in [7] the composition map $c \circ d^{-1}$ is a $C^{1+\alpha^-}$ smooth map if and only if the cylinder structures of the canonical charts c and d are $(1+\alpha)$-scale equivalent and $(1+\alpha)$-connection equivalent. Similarly to the proof of theorem 13, the $(1+\alpha)$-scale equivalence and $(1+\alpha)$-connection equivalence are equivalent to the following α-solenoid equivalence defined by equation (9).

For all $n \geq 1$ and for all $(w_n, v_n) \in \underline{\Omega}_{c,n} \cap \underline{\Omega}_{d,n}$ let $(\overline{w} = \ldots w_n \ldots w_1, \overline{v} = \ldots v_n \ldots v_1) \in \mathcal{S}$. By the β-pseudo-Hölder continuity of the solenoid function, for all $0 < \beta < \alpha$ there are constants $c, c_\beta > 0$ such that

$$\begin{aligned} \frac{s_c(w_n, v_n)}{s_d(w_n, v_n)} &\in \frac{s(\overline{w}, \overline{v})(1 \pm c_\beta(|C_{w_n} + C_{v_n}|))}{s(\overline{w}, \overline{v})(1 \pm c_\beta(|C_{w_n} + C_{v_n}|))} \\ &\subset 1 \pm c_\beta(|C_{w_n} + C_{v_n}|) \end{aligned} \tag{9}$$

and $s_c(w_n, v_n) > c$. Therefore, the cylinder structures of the canonical charts c and d are α-solenoid equivalent. ∎

Proof of theorem 17. Let $F : \Sigma_F \to \Sigma_F$ be the topological Markov map corresponding to the symbolic solenoid set $\mathcal{S}$. By the turntable condition of the solenoid function and by theorem 16, the canonical charts give a smooth structure of the set Σ_F.

For all $c = (\overline{t}, \overline{s}) \in \mathcal{S}_{GC}$ define $e = (\overline{v}, \overline{z}) \in \mathcal{S}_{GC}$ by $F(C_{t_n}) \subset F(C_{v_{n-1}})$ and $F(C_{s_n}) \subset F(C_{z_{n-1}})$, for all $n > 1$. By the construction of the canonical charts $c : \Sigma_c \to \mathbf{R}^+$ and $e : \Sigma_e \to \mathbf{R}^+$ the Markov map F is an affine map in Σ_c with respect to the charts c and e. Therefore, the Markov map F is a C^{1+} Markov map F. By construction of the canonical charts the solenoid function $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+ = \mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ ∎

Proof of theorem 18. By theorem 17 and theorem 16. ∎

13 Proof of theorems 6 and 8.

Proof of theorem 6. By theorem 5 and theorem 17 a C^{1+} Markov map F defines a solenoid function s_F and vice-versa. By theorem 1, if the C^r Markov maps F and G are C^{1+} conjugate then they are C^r-conjugate . Therefore, we have to prove that the smooth Markov maps F and G are C^{1+} conjugate if and only if the solenoid function $\mathrm{s}_F : \mathcal{S}_F \to \mathbf{R}^+$ is equal to the solenoid function $\mathrm{s}_G : \mathcal{S}_G \to \mathbf{R}^+$.

By theorem 15, the smooth Markov maps F and G are C^{1+} conjugate if and only if the cylinder structures of F and G are solenoid equivalent. Therefore, we will prove that the cylinder structures of F and G are solenoid equivalent if and only if $\mathrm{s}_F = \mathrm{s}_G$.

By smoothness of the Markov maps F and G and by theorem 14 the cylinder structures of F and G have the solenoid property.

Let us prove that if the cylinder structures of F and G are solenoid equivalent then the solenoid function $\mathrm{s}_F : \mathcal{S}_F \to \mathbf{R}^+$ is equal to the solenoid function $\mathrm{s}_G : \mathcal{S}_G \to \mathbf{R}^+$. Since the Markov maps F and G are topologically conjugate the solenoid set $\mathcal{S}_F$ and $\mathcal{S}_G$ are equal.

Since the cylinder structures of F and G are solenoid equivalent and have the solenoid property, there is $0 < \lambda < 1$ such that for all $(\overline{t} = \ldots t_1, \overline{s} = \ldots s_1) \in \mathcal{S}$ and for all $n \geq N_{\overline{t},\overline{s}} \geq 1$

$$\begin{aligned}
\frac{\mathrm{s}_F(\overline{t},\overline{s})}{\mathrm{s}_G(\overline{t},\overline{s})} &= \frac{\mathrm{s}_F(\overline{t},\overline{s})}{\mathrm{s}_F(t_n,s_n)} \frac{\mathrm{s}_F(t_n,s_n)}{\mathrm{s}_G(t_n,s_n)} \frac{\mathrm{s}_G(t_n,s_n)}{\mathrm{s}_G(\overline{t},\overline{s})} \\
&\in (1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n)) \\
&\subset 1 \pm \mathcal{O}(\lambda^n).
\end{aligned}$$

On letting n converge to infinity, we obtain that solenoid functions $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ and $\mathrm{s}_G : \mathcal{S} \to \mathbf{R}^+$ are equal.

Now, we prove that if the solenoid functions $\mathrm{s}_F : \mathcal{S} \to \mathbf{R}^+$ and $\mathrm{s}_G : \mathcal{S} \to \mathbf{R}^+$ are equal then the cylinder structures of F and G are solenoid equivalent.

For all $(t_n, s_n) \in \underline{\Omega}_n^s \times \underline{\Omega}_n^g$, choose $(\overline{t} = \ldots t_n \ldots t_1, \overline{s} = \ldots s_n \ldots s_1) \in \mathcal{S}$ such that $N_{\overline{t},\overline{s}} \leq n$. Since the cylinder structures of F and G have the solenoid property and

$\mathrm{s}_F(\bar{t},\bar{s}) = \mathrm{s}_G(\bar{t},\bar{s})$

$$\begin{aligned}
\frac{\mathrm{s}_F(t_n,s_n)}{\mathrm{s}_G(t_n,s_n)} &= \frac{\mathrm{s}_F(t_n,s_n)}{\mathrm{s}_F(\bar{t},\bar{s})}\frac{\mathrm{s}_F(\bar{t},\bar{s})}{\mathrm{s}_G(\bar{t},\bar{s})}\frac{\mathrm{s}_G(\bar{t},\bar{s})}{\mathrm{s}_G(t_n,s_n)} \\
&\in (1 \pm \mathcal{O}(\lambda^n))(1 \pm \mathcal{O}(\lambda^n)) \\
&\subset 1 \pm \mathcal{O}(\lambda^n).
\end{aligned}$$

Therefore, the cylinder structures of F and G are solenoid equivalent. ∎

Proof of lemma 4. Let F and G be two C^{1+} Markov maps such that $\mathrm{s}_F = \mathrm{s}_G = \mathrm{s}$. By theorem 6, the Markov maps F and G are C^{1+} conjugate. Thus, for all $t \in \Omega_F = \Omega_G$ $|C_t^F| = \mathcal{O}(|C_t^G|)$. Therefore, for all $0 < \beta < \alpha$ if the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is β pseudo-Hölder continuous with respect to the metric d defined by using the Markov map F then the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}^+$ is β pseudo-Hölder continuous with respect to the metric d defined by using the Markov map G. ∎

Proof of theorem 8. By theorem 7 and theorem 18 a $C^{1+\alpha^-}$ Markov map F defines an α-solenoid function s_F and vice-versa. By theorem 6, the smooth Markov maps F and G are C^{1+} conjugate by h if and only if the solenoid functions $\mathrm{s}_F : \mathcal{S}^F \to \mathbf{R}^+$ and $\mathrm{s}_G : \mathcal{S}^G \to \mathbf{R}^+$ are equal. By theorem 1, the conjugacy h is a $C^{1+\alpha^-}$ smooth map h. ∎

Proof of corollary 2. Let $\eta > \alpha$ and suppose that the Markov map F is $C^{1+\eta}$ smooth. By theorem 8, the solenoid function $\mathrm{s} : \mathcal{S} \to \mathbf{R}$ is η pseudo-Hölder continuous which is absurd. ∎

14 Proof of theorems 9, 10, 11 and 12.

Proof of theorem 9. By theorem 5, the C^{1+} Markov map F defines a solenoid function s_F. Let us check that the solenoid function s_F is α-determined.

By theorem 13, the cylinder structure of F has the α-solenoid property. Thus, for all $0 < \beta < \alpha$, for all $(\bar{t},\bar{s}) \in \mathcal{S}$ and for all $n > N_{\bar{t},\bar{s}}$,

$$\left|1 - \frac{\mathrm{s}(t_n,s_n)}{\mathrm{s}(t_{n-1},s_{n-1})}\right| \le \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^\beta).$$

By expansiveness of F, for all $p,q \ge n > N_{\bar{t},\bar{s}}$

$$\frac{\mathrm{s}(t_p,s_p)}{\mathrm{s}(t_q,s_q)} \in 1 \pm \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^\beta).$$

Therefore,

$$\frac{\mathrm{s}(\bar{t},\bar{s})}{\mathrm{s}(t_n,s_n)} \in 1 \pm \mathcal{O}((|C_{t_n}| + |C_{s_n}|)^\beta).$$

The solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is α-determined. ∎

Proof of theorem 10. First, we prove that the cylinder structure corresponding to the pre-solenoid function $s = s_F : \underline{\Omega} \to \mathbf{R}^+$ has the α-solenoid property.

The cylinder structure of G has bounded geometry by compactness of the set $\mathcal{S}_{GC}$ which proves condition (i) of α-solenoid property.

For all $0 < \beta < \alpha$ and for all $(t_n, s_n) \in \underline{\Omega}_n^s \cup \underline{\Omega}_n^g$, let $(\overline{t} = \ldots t_n \ldots t_1, \overline{s} = \ldots s_n \ldots s_1) \in \mathcal{S}$. By the α-determined property of the solenoid function condition (ii) is also verified

$$\begin{aligned} \frac{s(s_n, t_n)}{s(s_{n-1}, t_{n-1})} &\in \frac{s(\overline{s}, \overline{t})(1 \pm \mathcal{O}((|C_{s_n}| + |C_{t_n}|)^\beta))}{s(\overline{s}, \overline{t})(1 \pm \mathcal{O}((|C_{s_{n-1}}| + |C_{t_{n-1}}|)^\beta))} \\ &\subset 1 \pm \mathcal{O}((|C_{s_n}| + |C_{t_n}|)^\beta). \end{aligned}$$

By theorem 13, there is a $C^{1+\alpha^-}$ Markov map G with pre-solenoid function $s_G = s_F : \underline{\Omega} \to \mathbf{R}^+$. ∎

Proof of theorem 11. By theorem 5, the Markov map F defines a solenoid function s_F. Let us check that the solenoid function s_F is determined.

By theorem 14, the cylinder structure of F has the solenoid property. Thus, there is $0 < \lambda < 1$ such that for all $\overline{t}, \overline{s} \in \mathcal{S}$ and for all $n > N_{\overline{t},\overline{s}}$,

$$\left| 1 - \frac{s(t_n, s_n)}{s(t_{n-1}, s_{n-1})} \right| \le \mathcal{O}(\lambda^n).$$

Therefore,

$$\frac{s(\overline{t}, \overline{s})}{s(t_n, s_n)} \in 1 \pm \mathcal{O}(\lambda^n).$$

The solenoid function $s_F : \mathcal{S} \to \mathbf{R}^+$ is determined. ∎

Proof of theorem 12. First, we prove that the cylinder structure corresponding to the pre-solenoid function $s = s_F : \underline{\Omega} \to \mathbf{R}^+$ has the solenoid property.

The cylinder structure of G has bounded geometry by compactness of the set $\mathcal{S}_{GC}$ which proves condition (i) of the solenoid property.

For all $(t_n, s_n) \in \underline{\Omega}_n^s \times \underline{\Omega}_n^g$, let $(\overline{t} = \ldots t_n \ldots t_1, \overline{s} = \ldots s_n \ldots s_1) \in \mathcal{S}$. By the determined property of the solenoid function condition (ii) is also verified

$$\begin{aligned} \frac{s(s_n, t_n)}{s(s_{n-1}, t_{n-1})} &\in \frac{s(\overline{s}, \overline{t})(1 \pm \mathcal{O}(\lambda^n))}{s(\overline{s}, \overline{t})(1 \pm \mathcal{O}(\lambda^{n-1}))} \\ &\subset 1 \pm \mathcal{O}(\lambda^n). \end{aligned}$$

By theorem 14, there is a C^{1+} Markov map G with pre-solenoid function $s_G = s_F : \underline{\Omega} \to \mathbf{R}^+$. ∎

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