

## NASH AND SOCIAL WELFARE IMPACT IN AN INTERNATIONAL TRADE MODEL

FILIFE MARTINS

Department of Mathematics and LIAAD-INESC  
Faculty of Sciences, University of Porto  
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

ALBERTO A. PINTO

Department of Mathematics and LIAAD-INESC  
Faculty of Sciences, University of Porto  
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

JORGE PASSAMANI ZUBELLI

IMPA  
Instituto Nacional de Matemática Pura e Aplicada  
Estrada Dona Castorina, 110, RJ 22460-320, Rio de Janeiro, Brasil

(Communicated by Michel Benaïm)

**ABSTRACT.** We study a classic international trade model consisting of a strategic game in the tariffs of the governments. The model is a two-stage game where, at the first stage, governments of each country use their welfare functions to choose their tariffs either (i) competitively (Nash equilibrium) or (ii) cooperatively (social optimum). In the second stage, firms choose competitively (Nash) their home and export quantities. We compare the competitive (Nash) tariffs with the cooperative (social) tariffs and we classify the game type according to the coincidence or not of these equilibria as a social equilibrium, a prisoner's dilemma or a lose-win dilemma.

**1. Introduction.** In this work, we consider a classic duopoly international trade model with complete information, where there are two countries and a firm in each country that sells in its own country and exports to the other one. The exportation is subject to a tariff fixed by the government of the importing country. The international trade model has two stages: in the first stage, the governments simultaneously choose their tariff rates; and in the second stage, the firms observe the tariff rates and simultaneously choose their quantities for home consumption and for export (see, for instance, [20]).

For the second stage, we consider always the classic competitive (Nash) equilibria that determines uniquely the quantities for home consumption and for export. Now, for the first stage, the decision of the governments to impose or not tariffs can be interpreted as the actions of a game specified by the utilities considered for each country. The utilities (each corresponding to a different game) of the countries that

---

2010 *Mathematics Subject Classification.* Primary: 91B14, 91B15, 91B60; Secondary: 91B64.

*Key words and phrases.* International trade, international duopoly, tariffs game; prisoner's dilemma, Nash equilibrium; social optimum, welfare.

we analyze are the relevant economic quantities of the international trade model for the consumers and firms. In particular, we consider the utilities given by the custom revenue of the countries, the consumer surplus of each country, the profit of the firms and the welfare of the countries at the competitive Nash equilibrium of the second stage game. We show that for each of the above utilities there is a Nash (competitive) equilibrium and a social optimum equilibrium corresponding to the maximization of the joint utility of the two governments. For each one of the utilities, we will compare these social and Nash equilibria in terms of the economic relevant quantities of the model.

There are three typical game outcomes: the social equilibrium (**SE**), where the social optimum coincides with the Nash equilibrium; the prisoner's dilemma (**PD**), where both utilities are bigger in the social optimum than in the Nash equilibrium; and the lose-win social strategies (**LW**), where the utilities of one of the countries is bigger in the social optimum tariffs and utility of the other country is bigger in the Nash equilibrium tariffs. For every pair of utilities, we will find which of the three types of outcomes **SE**, **PD** or **LW** occurs in terms of the model parameters.

Which one of the three previously mentioned outcomes (**SE**, **PD** or **LW**) occurs presents qualitatively different scenarios for the involved countries. If the game is of the social equilibrium (**SE**) type, then there is a priori no need of a trade agreement, because the two countries are already in the social optimum as the competitive equilibrium coincides with the social equilibrium that maximizes the joint utilities of the two countries. If the game is of prisoner's dilemma (**PD**) type or of lose-win social strategies (**LW**) type, then at least one of the countries can improve its payoff if they choose to cooperate, and that can be done by means of a trade agreement. In the first case, both governments can make a trade agreement such that they choose the social tariffs, thereby improving the utilities of the two countries. In the second case, both governments can make a trade agreement such that both countries opt for the social tariffs. However, in this case, the situation is qualitatively different as one of the countries is injured by the change to the social tariffs. So in order to enforce cooperation there is need to compensate that country, for instance, by means of a financial compensation or in other terms stated in the agreement.

There is a vast literature in international trade models with both complete and incomplete information (see, for instance, [4, 6, 7, 8, 9, 11, 12, 19, 22, 25, 26, 31]). In [34], the authors proposed a model including government subsidies to firms in the form of R&D subsidies. In [5], they studied a model where governments subsidize firms over the produced quantities and also study the extension of the model to a supra-game between governments. The relation with the tariffs literature is that when a negative optimal level of subsidy occurs, it is interpreted as an export tariff (see also [21]). The question of the enforcement of trade agreements, i.e., the enforcement of cooperating strategies, is also a very active research topic (see a review of early contributions to this topic in [35]). The enforcement of trade agreements, i.e., the enforcement of cooperating strategies has two important features. Firstly, is that at least one country may have an incentive to unilaterally deviate from its social (cooperative) tariff to its Nash (competitive) tariff. Consequently, such country will eventually deviate to its competitive tariff if there is no punishment associated to this. Therefore, it is important that trade agreements present a mechanism to punish such deviations. Secondly, one must observe that there is

not a supra-national authority to enact the punishment mechanism of the violating country for its eventual deviations from cooperation. This implies the need for international agreements to be self-enforcing.

These characteristics, particularly that of self-enforcing agreements lead to the study of the enforcement issues by means of certain repeated games that possess and present a good deal of important features of cooperative self-enforcing agreements (see, for example, [1, 2, 10]). As a result, several instruments have been proposed and studied with the objective of achieving the (self)-enforcement of international trade agreements, as well as addressing the question of their efficiency, particularly when compared to the threat of tariff retaliation. In [27], the authors adopted the repeated game approach and proposed alternative instruments in the context of a trade agreement between two symmetric countries. More precisely, the authors compared the effectiveness of retaliatory tariffs with that of a financial compensation by means of a monetary fine to the country that violates cooperation. Unlike the most common tariff retaliation, which is imposed by the injured country and only depends on it, monetary fines have an enforcing problem because they must be voluntarily paid by the country that has deviated from cooperation. They showed that monetary fines yield the same cooperation as tariff retaliation, except when a country deviated from the agreement due to an unanticipated shock in the model parameters (in the case, a political economy parameter) with monetary fines being preferable to tariff retaliation in that situation. They also studied the possibility of the countries exchanging bonds, and concluded that this yields the same outcome as tariff retaliation. In [28], the same authors introduced size inequality between countries, by considering one large country and a region of equal market size with a number of identical small countries. The fact that in the second region countries are individually small generates a coordination externality among themselves, as they cannot credibly threat tariff retaliation, but they would if they act like a group. Thus, in trade agreements based solely on tariff retaliation, coordination externalities generate asymmetric outcomes. They showed that improvements in efficiency and more symmetric outcomes can be obtained by including specific financial instruments such as monetary fines and bonds in conventional trade agreements based only on the threat of tariff retaliation. In [24], a model with two asymmetric countries is considered and it is shown that an efficient trade agreement might not lead to free trade. Various types of transfers between countries are studied, such as financial (monetary fines), foreign aid and side payments intended to offset a loss resulting from a trade agreement.

Within the frame of repeated games, in [3, 29], the authors considered a model with two firms competing in the same country and they study the phenomena of dumping. They interpreted the deviation from cooperation by the foreign firm as dumping, with the foreign firm subsequently suffering a period of punishment, where punishment results from the home firm lobby on its government, and the imposition of a tariff that makes the foreign country unable to export during the period of punishment. In [30], the authors considered a model where a firm has the monopoly in its home market, but divides the market of the foreign country with a firm from that country. They studied deviation from cooperation in the foreign market by the firm that sells in the two countries either by increasing production solely in its foreign market and committing dumping, or by increasing production in both countries, and deviate without making dumping.

This paper is structured in the following way. In section 2 we present the fundamental concepts of Nash and Social tariffs and the comparison between these two equilibria and the type of game that is obtained according to this comparison. In section 3 we present the international duopoly model and the most relevant economic quantities of the model. In section 4 we solve the second stage game between the firms. In section 5 we compute the Nash and Social tariffs for each relevant economic quantity of the duopoly model, considering each one as the utility function of the countries. We compare such tariffs and classify the games according to their types. In section 6 we focus on the case of the welfare of the two countries, showing that the outcome is either of lose-win type (**LW**) or prisoner's dilemma type (**PD**). We present a full characterization of the game outcomes in terms of the tax-free home production indexes (see figure 1). We present the conclusions of the paper in section 7.

**2. Strategic tariffs.** In this section, we introduce the most relevant game theoretical concepts that we will use in the other sections to understand the strategic behaviour of firms, consumers and governments of the countries.

Let  $u_i(t_i, t_j)$  and  $u_j(t_i, t_j)$  be two relevant economic quantities of the countries  $X_i$  and  $X_j$  depending only upon the tariffs  $t_i$  and  $t_j$  imposed by the governments of the two countries. For instance, for every pair of tariffs  $(t_i, t_j)$ , the functions  $u_i(t_i, t_j)$  and  $u_j(t_i, t_j)$  can be the profit of the firms or the consumer surplus at the competitive Nash equilibrium for the quantities produced by the firms.

We are going to interpret  $u_i(t_i, t_j)$  and  $u_j(t_i, t_j)$  as the utilities of a game where the players are the governments of the countries and their actions are the tariffs  $(t_i, t_j)$ .

The quantity  $t_i^{BR}(t_j) \equiv t_i^{BR}(t_j; u)$  is a *best response* of the country  $X_i$  for the utility  $u_i$ , if for all tariffs  $t_i$ ,

$$u_i(t_i^{BR}(t_j), t_j) \geq u_i(t_i, t_j) .$$

A pair of tariffs  $(t_i^N, t_j^N) \equiv (t_i^N(u), t_j^N(u))$  is a *Nash equilibrium* or a *global strategic optimum*, if for all tariffs  $t_i$

$$u_i(t_i^N, t_j^N) \geq u_i(t_i, t_j^N) ,$$

and for all tariffs  $t_j$

$$u_j(t_i^N, t_j^N) \geq u_j(t_i^N, t_j) .$$

In other words, a pair of tariffs  $(t_i^N, t_j^N)$  is a Nash equilibrium, if

$$t_i^N = t_i^{BR}(t_j^N) \quad \text{and} \quad t_j^N = t_j^{BR}(t_i^N) .$$

A pair of tariffs  $(t_i^P, t_j^P) \equiv (t_i^P(u), t_j^P(u))$  is a *Pareto optimum*, if there is no pair  $(t_i, t_j)$  of tariffs such that

$$u_i(t_i, t_j) \geq u_i(t_i^P, t_j^P) \quad \text{for all } i, j \in \{1, 2\},$$

and at least one utility  $u_i$ ,  $i \in \{1, 2\}$  gets a better payoff with  $(t_i, t_j)$  than with  $(t_i^P, t_j^P)$ , i.e.

$$u_i(t_i, t_j) > u_i(t_i^P, t_j^P) .$$

The *social utility* (or *total utility*)  $u_T$  is

$$u_T(t_i, t_j) = u_i(t_i, t_j) + u_j(t_i, t_j) .$$

The quantity  $t_i^{SR}(t_j) \equiv t_i^{SR}(t_j; u)$  is a *social best response*, if for all tariffs  $t_i$

$$u_T(t_i^{SR}(t_j), t_j) \geq u_T(t_i, t_j) .$$

A pair of tariffs  $(t_i^S, t_j^S) \equiv (t_i^S(u), t_j^S(u))$  is a *social optimum*, if for all tariffs  $t_i$

$$u_T(t_i^S, t_j^S) \geq u_T(t_i, t_j^S) ,$$

and for all tariffs  $t_j$

$$u_T(t_i^S, t_j^S) \geq u_T(t_i^S, t_j) .$$

In other words, a pair of tariffs  $(t_i^S, t_j^S)$  is a social optimum, if

$$t_i^S = t_i^{SR}(t_j^S) \quad \text{and} \quad t_j^S = t_j^{SR}(t_i^S) .$$

We observe that a social optimum is a Pareto optimum. For games with a unique Nash equilibrium, we describe the three typical games outcomes when we compare the social optimum with the Nash equilibrium.

**(SE) Social equilibrium:** When the social optimum coincides with the Nash equilibrium

$$(t_i^S, t_j^S) = (t_i^N, t_j^N)$$

and the social optimum is the only Pareto optimum. In this case, the individualist Nash choice of the tariffs by the governments leads to a social equilibrium. Hence, a priori there is no need of a trade agreement between the two governments of the two countries.

**(PD) Prisoner's dilemma:** When the social optimum  $(t_i^S, t_j^S)$  is different from the Nash equilibrium

$$t_i^S \neq t_i^N \quad \text{or} \quad t_j^S \neq t_j^N$$

and both utilities are bigger in the social optimum than in the Nash equilibrium,

$$u_i(t_i^S, t_j^S) > u_i(t_i^N, t_j^N) \quad \text{and} \quad u_j(t_i^S, t_j^S) > u_j(t_i^N, t_j^N) .$$

In this case, the game is like the Prisoner's dilemma, where the Nash strategy leads to a lower outcome for both countries than if they would agree among therein (through a trade agreement) in opting for the social optimum. When we obtain

$$u_i(t_i^S, t_j^S) = u_i(t_i^N, t_j^N) \quad \text{and} \quad u_j(t_i^S, t_j^S) > u_j(t_i^N, t_j^N) .$$

or

$$u_i(t_i^S, t_j^S) > u_i(t_i^N, t_j^N) \quad \text{and} \quad u_j(t_i^S, t_j^S) = u_j(t_i^N, t_j^N) .$$

we say the game is a (weak) Prisoner's dilemma.

**(LW) Lose-win social strategies:** When the social optimum  $(t_i^S, t_j^S)$  is different from the Nash equilibrium

$$t_i^S \neq t_i^N \quad \text{or} \quad t_j^S \neq t_j^N$$

and one of the utilities is bigger in the social optimum and the other utility is bigger in the Nash equilibrium, i.e.,

$$u_i(t_i^S, t_j^S) < u_i(t_i^N, t_j^N) \quad \text{and} \quad u_j(t_i^S, t_j^S) > u_j(t_i^N, t_j^N) ,$$

or

$$u_i(t_i^S, t_j^S) > u_i(t_i^N, t_j^N) \quad \text{and} \quad u_j(t_i^S, t_j^S) < u_j(t_i^N, t_j^N) .$$

When the game is of lose-win type there are two possible outcomes as described above. We will denote such outcomes respectively by  $\mathbf{L}_i\mathbf{W}_j$  and  $\mathbf{L}_j\mathbf{W}_i$ . The first indicates that the country  $X_i$  has a utility loss and country  $X_j$  has a utility gain while enforcing the social optimum, and the second indicates the opposite situation.

In this case, the governments can implement an external mechanism (trade agreement) that will make them to opt for the social optimum in such a way that the country that gets an advantage in its utility compensates the loss in the utility of the other country and can also give some extra benefit in order to persuade the other country to implement the social equilibrium.

**3. International duopoly model.** In this section, we introduce the relevant economic quantities of the international duopoly model.

The international duopoly model is a game with two stages (sub-games). In the first stage, both governments choose simultaneously their Nash or social tariffs for a utility given by a relevant economic quantity; and, in the second stage, the firms choose simultaneously their home and export quantities to maximize competitively their profits.

The *home consumption*  $h_i$  is the quantity produced by the firm  $F_i$  and consumed in its own country  $X_i$ . The *export*  $e_i$  is the quantity produced by the firm  $F_i$  and consumed in the country  $X_j$  of the other firm  $F_j$ , where  $i, j \in \{1, 2\}$  with  $i \neq j$ . The *tariff rate*  $t_i$  is determined by the government of country  $X_i$  on the import quantity  $e_j$ . The *total quantity*  $q_i$  produced by firm  $F_i$  is

$$q_i \equiv q_i(h_i, e_i) = h_i + e_i .$$

The *aggregate quantity*  $Q_i$  sold on the market in the country  $X_i$  is

$$Q_i \equiv Q_i(h_i, e_j) = h_i + e_j .$$

The *inverse demand*  $p_i$  in the country  $X_i$  is

$$p_i \equiv p_i(h_i, e_j) = \alpha_i - Q_i ,$$

where  $\alpha_i$  is the *demand intercept* of country  $X_i$ .

The *payoff*  $\pi_i$  of firm  $F_i$  is

$$\pi_i \equiv \pi_i(h_i, e_i, h_j, e_j; t_i, t_j) = (p_i - c_i)h_i + (p_j - c_i)e_i - t_j e_i ,$$

where  $c_i \geq 0$  is the firm  $F_i$ 's *unitary production cost* such that  $\alpha_i - c_i > 0$ , and  $t_j \geq 0$  is the tariff fixed by the government of country  $X_j$ .

The *custom revenue*  $CR_i$  of the country  $X_i$  is given by

$$CR_i \equiv CR_i(e_j; t_i) = t_i e_j .$$

The *consumer surplus*  $CS_i$  in the country  $X_i$  is given by

$$CS_i \equiv CS_i(h_i, e_j) = \frac{1}{2} Q_i^2 .$$

The *welfare*  $W_i$  of the country  $X_i$  is

$$W_i \equiv W_i(h_i, e_i, h_j, e_j; t_i, t_j) = CR_i + CS_i + \pi_i .$$

4. **Second stage Nash equilibrium.** In this section, we give a presentation of the well-known Nash equilibrium of the second sub-game, i.e., firms choose the home and export quantities that competitively maximize their profits, in the case of complete information, i.e. when both firms have full information on their and others utility functions.

Let  $i, j \in \{1, 2\}$  with  $i \neq j$ . Define

$$T_i \equiv T_i(c_i, c_j) = (\alpha_i + c_i - 2c_j)/2 ,$$

$$T_j \equiv T_j(c_i, c_j) = (\alpha_j + c_j - 2c_i)/2 .$$

We also define

$$T_i^* \equiv T_i^*(c_i, c_j) = (\alpha_i + c_j - 2c_i)/2 ,$$

$$T_j^* \equiv T_j^*(c_i, c_j) = (\alpha_j + c_i - 2c_j)/2 .$$

Denoting  $\Delta\alpha := \alpha_i - \alpha_j$ , we have that

$$T_i^* = T_j + \Delta\alpha/2 ,$$

$$T_j^* = T_i - \Delta\alpha/2 .$$

This yields

$$T_i + T_j = T_i^* + T_j^* . \tag{4.1}$$

We also have that

$$\alpha_i - c_i = \frac{2(T_i + 2T_i^*)}{3} ,$$

$$\alpha_j - c_j = \frac{2(T_j + 2T_j^*)}{3} .$$

Denoting  $\Delta c := c_i - c_j$  we have

$$\Delta c = \frac{2(T_j^* - T_j)}{3} = \frac{2(T_i - T_i^*)}{3} .$$

**Assumption (A).**  $T_i > 0$ ,  $T_j > 0$ ,  $T_i^* > 0$  and  $T_j^* > 0$ .

We observe that under assumption (A),  $\alpha_i - c_i > 0$  and  $\alpha_j - c_j > 0$ .

The *best response*  $(h_i^{BR}(e_j), e_i^{BR}(h_j; t_j))$  of the firm  $F_i$  is the solution of

$$(h_i^{BR}(e_j), e_i^{BR}(h_j; t_j)) = \arg \max_{(h_i, e_i)} \pi_i(h_i, e_i, h_j, e_j; t_i, t_j) .$$

Hence

$$\begin{cases} h_i^{BR}(e_j) = \frac{\alpha_i - e_j - c_i}{2} \\ e_i^{BR}(h_j; t_j) = \frac{\alpha_j - h_j - c_i - t_j}{2} . \end{cases}$$

The *Nash equilibrium*  $(h_i(t_i), e_i(t_j); h_j(t_j), e_j(t_i))$  is the solution of

$$\begin{cases} (h_i(t_i), e_i(t_j)) = (h_i^{BR}(e_j(t_i)), e_i^{BR}(h_j(t_j); t_j)) \\ (h_j(t_j), e_j(t_i)) = (h_j^{BR}(e_i(t_j)), e_j^{BR}(h_i(t_i); t_i)) . \end{cases}$$

So, for every  $t_i \in [0, T_i]$  and every  $t_j \in [0, T_j]$ , the home  $h_i(t_i)$  and export  $e_i(t_j)$  quantities for the firms at the Nash equilibrium (see [20]) are

$$\begin{aligned}
h_i(t_i) &\equiv h_i(c_i, c_j; t_i) = \frac{2T_i^* + t_i}{3} \\
e_i(t_j) &\equiv e_i(c_i, c_j; t_j) = \frac{2(T_j - t_j)}{3} \\
h_j(t_j) &\equiv h_j(c_i, c_j; t_i) = \frac{2T_j^* + t_j}{3} \\
e_j(t_i) &\equiv e_j(c_i, c_j; t_j) = \frac{2(T_i - t_i)}{3}
\end{aligned}$$

We note that an increase in the quantities  $T_i^*$  and  $T_j^*$  generates an increase in the home quantities produced by the firms at a given tariff level. The quantities  $T_i^*$  and  $T_j^*$  have a clear economic interpretation. An increase in quantity  $T_i^*$  is good for firm  $F_i$  since this can occur due to three different possibilities, all of whom favour firm  $F_i$ : an increase in the home market size  $\alpha_i$  of country  $X_i$ ; an increase in the production costs of the opponent firm  $c_j$ ; a decrease in firm  $i$  own production cost  $c_i$ . So any of these three situations generates an increase in home quantities, albeit at different rates: at a given tariff level, a reduction in own costs increases home production at twice the speed as an increase in the home market size or a decrease in the opponent's costs.

Since the export quantities are non-negative,  $T_i$  and  $T_j$  are the *maximal tariffs*. When country  $X_i$  is tax-free, then firm  $F_j$  exports  $2/3$  of the admissible maximal tariff  $T_i$ . The maximal tariff  $T_i$  can increase due to several reasons: an increase in the market size  $\alpha_i$  of country  $X_i$ ; an increase in costs  $c_i$  of firm  $F_i$ ; a decrease in the costs  $c_j$  of firm  $F_j$ . We further observe that at a fixed tariff rate, a decrease in the production cost  $c_j$  of firm  $F_j$  increases its export quantity faster, as the rate of change in the maximal tariff  $T_i$  due to a decrease in  $c_j$  is twice as big as the rate of change due to an increase in the market size of country  $X_i$  or a competitive loss in firm  $F_i$  because of increased production costs.

We observe that the decrease in one country's export quantity due to the other country's tariff is twice the increment in the home quantity of the other country.

Let

$$R_i = \frac{T_i^*}{T_i} \text{ and } R_j = \frac{T_j^*}{T_j} .$$

We observe that under assumption (A),  $R_i > 0$  and  $R_j > 0$ . Using equality (4.1) we obtain

$$(1 - R_i)T_i = (R_j - 1)T_j .$$

Hence, the ratios  $R_i$  and  $R_j$  satisfy the relation

$$R_i < 1 \Leftrightarrow R_j > 1 .$$

Furthermore, we have that  $R_i = 1$  if and only if  $R_j = 1$ , meaning that for every  $T_i$  and  $T_j$ ,  $T_i^* = T_i$  and  $T_j^* = T_j$ . When  $R_i \neq 1$

$$\frac{T_i}{T_j} = \frac{(R_j - 1)}{(1 - R_i)} .$$

The *tax-free home production index* is

$$H_i = \frac{h_i^N(0)}{h_i^N(T_i)} = \frac{2T_i^*}{T_i + 2T_i^*} = \frac{2R_i}{1 + 2R_i} ,$$

where  $h_i^N(0)$  corresponds to the home production of country  $i$  when there are tax-free exports from country  $j$  to country  $i$ , and  $h_i^N(T_i)$  to the monopoly home production of country  $i$  when  $j$  does not export. We have

$$R_i = \frac{H_i}{2(1 - H_i)} .$$

Hence, the indexes  $H_i$  and  $H_j$  satisfy  $0 < H_i < 1$ ,  $0 < H_j < 1$  and the relation

$$0 < H_i < 2/3 \Leftrightarrow 2/3 < H_j < 1 .$$

Furthermore, we have that  $H_i = 2/3$  if and only if  $H_j = 2/3$ , meaning that for every  $T_i$  and  $T_j$ ,  $T_i^* = T_i$  and  $T_j^* = T_j$  (and  $R_i = 1 = R_j$ ). When  $H_i \neq 2/3$ ,

$$\frac{T_i}{T_j} = \frac{(H_i - 1)(3H_j - 2)}{(H_j - 1)(2 - 3H_i)} .$$

We observe that the country whose tax-free home production index is closer to 1 is the country that faces a lower decrease in their home production quantities while changing from a monopoly situation to a tax-free situation where the other country exports freely.

**5. Strategic games.** In this section, we will analyse the advantages and disadvantages of the use of tariffs for the firms, the consumers and the governments of the countries. To do it, we will use the relevant economic quantities as utilities  $u_i(t_i, t_j)$  and  $u_j(t_i, t_j)$  of a game where the players are the governments of the countries and their actions are the tariffs  $(t_i, t_j)$ . For each pair of utilities that we will consider, we will find which of the three typical games occurs: social equilibrium (**SE**), prisoner's dilemma (**PD**), or lose-win social strategies (**LW**).

**5.1. Tariff effects in produced quantities and prices.** We first consider the case where the utilities are the home quantities, i.e.,  $u_i = h_i$ . The home quantity  $h_i(t_i)$  increases with the tariff  $t_i$  and the home quantity  $h_j(t_j)$  increases with the tariff  $t_j$ , and so

$$t_i^{BR}(t_j; h) = T_i , \quad t_j^{BR}(t_i; h) = T_j \quad \text{and} \quad (t_i^N(h), t_j^N(h)) = (T_i, T_j) .$$

The social utility  $h_T(t_i, t_j)$  is

$$h_T(t_i, t_j) = \frac{2(T_i^* + T_j^*) + t_i + t_j}{3}$$

and so the social utility also increases with both tariffs  $t_i$  and  $t_j$ , so

$$t_i^{SR}(t_j; h) = T_i , \quad t_j^{SR}(t_i; h) = T_j \quad \text{and} \quad (t_i^S(h), t_j^S(h)) = (T_i, T_j) .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(h) = t_i^N(h) = T_i .$$

Therefore, the game with utility  $u_i = h_i$ , is of the type **SE**.

When we consider the utilities to be the export quantities, i.e.  $u_i = e_i$ , we see that the export quantity  $e_i(t_j)$  decreases with the tariff  $t_j$ , but does not depend upon tariff  $t_i$ . The same occurs with the export quantity  $e_j(t_i)$  that decreases with tariff  $t_i$ , but does not depend upon tariff  $t_j$ . Hence, every tariff  $t_i$  is a best response to any tariff  $t_j$ , and vice-versa, and so every pair of tariffs is a Nash equilibrium:

$$t_i^{BR}(t_j; e) \in [0, T_i] , \quad t_j^{BR}(t_i; e) \in [0, T_j] \quad \text{and} \quad (t_i^N(e), t_j^N(e)) \in [0, T_i] \times [0, T_j] .$$

The social utility  $e_T(t_i, t_j)$  is

$$e_T(t_i, t_j) = \frac{2(T_i + T_j) - 2(t_i + t_j)}{3}$$

and so

$$t_i^{SR}(t_j; e) = 0, \quad t_j^{SR}(t_i; e) = 0 \quad \text{and} \quad (t_i^S(e), t_j^S(e)) = (0, 0).$$

Hence, there is a unique social optimum, that is the unique Pareto optimum

$$t_i^S(e) = 0.$$

Therefore, for the game with utility  $u_i = e_i$ , we have:

1. For the Nash tariff  $(t_i^N, t_j^N) = (0, 0)$ , the game is of **SE** type;
2. For all other Nash tariffs the game is of **PD** type.

We now consider the case where the utilities are the total quantities produced by the firms, i.e.,  $u_i = q_i$ . The total quantity  $q_i(t_i, t_j)$  produced by firm  $F_i$  is given by

$$q_i(t_i, t_j) \equiv q_i(c_i, c_j; t_i, t_j) = \frac{1}{3}(2T_i^* + 2T_j + t_i - 2t_j)$$

and so the total quantity  $q_i^N(t_i, t_j)$  increases with  $t_i$  and decreases with  $t_j$ . For firm  $F_j$  we have

$$q_j(t_i, t_j) \equiv q_j(c_i, c_j; t_i, t_j) = \frac{1}{3}(2T_j^* + 2T_i + t_j - 2t_i)$$

so the total quantity increases with  $t_j$  and decreases with  $t_i$ . Thus, there is a unique Nash equilibrium

$$t_i^{BR}(t_j; q) = T_i, \quad t_j^{BR}(t_i; q) = T_j \quad \text{and} \quad (t_i^N(q), t_j^N(q)) = (T_i, T_j).$$

The social utility  $q_T(t_i, t_j)$  is

$$q_T(t_i, t_j) = \frac{4(T_i + T_j) - (t_i + t_j)}{3}$$

and so

$$t_i^{SR}(t_j; q) = 0, \quad t_j^{SR}(t_i; q) = 0 \quad \text{and} \quad (t_i^S(q), t_j^S(q)) = (0, 0).$$

Hence, there is a unique social optimum but it does not coincide with the Nash equilibrium

$$t_i^S(q) \neq t_i^N(q).$$

We have that

$$q_i(t_i^N, t_j^N) < q_i(t_i^S, t_j^S) \quad \text{if and only if} \quad T_i < 2T_j.$$

$$q_j(t_i^N, t_j^N) < q_j(t_i^S, t_j^S) \quad \text{if and only if} \quad T_j < 2T_i.$$

Hence, we have two possible cases:

**Case I.**  $T_i/2 \leq T_j \leq 2T_i$ . Then we have

$$q_i(t_i^N, t_j^N) \leq q_i(t_i^S, t_j^S) \quad \text{and} \quad q_j(t_i^N, t_j^N) \leq q_j(t_i^S, t_j^S).$$

Therefore, the game is of the type **PD**.

**Case II.**  $T_j < T_i/2$ . The case  $2T_i < T_j$  is similar. Then we have

$$q_i(t_i^N, t_j^N) > q_i(t_i^S, t_j^S) \quad \text{and} \quad q_j(t_i^N, t_j^N) < q_j(t_i^S, t_j^S) .$$

Therefore, the game is of the type **LW**. More precisely, the outcome is  $\mathbf{L}_i \mathbf{W}_j$ .

We now consider the utilities to be the aggregate quantities in each country, i.e.,  $u_i = Q_i$ . The aggregate quantity  $Q_i^N(t_i)$  in the market of country  $X_i$  is

$$Q_i(t_i) \equiv Q_i(c_i, c_j; t_i) = \frac{2(T_i + T_i^*) - t_i}{3} ,$$

and the aggregate quantity  $Q_j(t_j)$  in the market of country  $X_j$  is

$$Q_j(t_j) \equiv Q_j(c_i, c_j; t_j) = \frac{2(T_j + T_j^*) - t_j}{3} ,$$

and so the aggregate quantities decrease with the respective tariffs. So, we have

$$t_i^{BR}(t_j; Q) = 0 \quad , \quad t_j^{BR}(t_i; Q) = 0 \quad \text{and} \quad (t_i^N(Q), t_j^N(Q)) = (0, 0) .$$

The social utility  $Q_T(t_i, t_j)$  is

$$Q_T(t_i, t_j) = q_T(t_i, t_j)$$

and so

$$(t_i^S(Q), t_j^S(Q)) = (0, 0) .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(Q) = t_i^N(Q) = 0 .$$

Therefore, the game with utility  $u_i = Q_i$ , is of the type **SE**.

We now consider the utility of the countries to be the symmetric of the prices, i.e.,  $u_i = p_i$ . The inverse demand function  $p_i(t_i)$  in the country  $X_i$  is

$$p_i(t_i) \equiv p_i(c_i, c_j; t_i) = \alpha_i - \frac{2(T_i^* + T_i) - t_i}{3}$$

and the inverse demand function  $p_j(t_j)$  in the country  $X_j$  is

$$p_j(t_j) \equiv p_j(c_i, c_j; t_j) = \alpha_j - \frac{2(T_j^* + T_j) - t_j}{3} ,$$

and so the inverse demand functions of the two countries increase with the respective tariffs. So we have

$$t_i^{BR}(t_j; p) = T_i \quad , \quad t_j^{BR}(t_i; p) = T_j \quad \text{and} \quad (t_i^N(p), t_j^N(p)) = (T_i, T_j) .$$

The social utility  $p_T(t_i, t_j)$  is

$$p_T(t_i, t_j) = \alpha_i + \alpha_j - \frac{2(T_i + T_j + T_i^* + T_j^*) - (t_i + t_j)}{3}$$

and so

$$t_i^{SR}(t_j; p) = T_i \quad , \quad t_j^{SR}(t_i; p) = T_j \quad \text{and} \quad (t_i^S(p), t_j^S(p)) = (T_i, T_j) .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(p) = t_i^N(p) = T_i .$$

Therefore, the game with utility  $u_i = p_i$ , is of the type **SE**. When the utility is  $u_i = -p_i$  then the solution is the same as the aggregate quantity in the market of country  $X_i$ , yielding a **SE** type game with equilibrium  $(0, 0)$ .

**5.2. Governments direct gains from using tariffs.** We now analyse the government's direct gains from using tariffs, i.e., the case where the utilities are given by the custom revenues,  $u_i = CR_i$ .

The custom revenue  $CR_i(t_i)$  of country  $X_i$  is given by

$$CR_i(t_i) \equiv CR_i(c_i, c_j; t_i) = \frac{2t_i(T_i - t_i)}{3},$$

and the custom revenue  $CR_j(t_j)$  of country  $X_j$  is given by

$$CR_j(t_j) \equiv CR_j(c_i, c_j; t_j) = \frac{2t_j(T_j - t_j)}{3}.$$

We have that  $CR_i(t_i) \geq 0$  and  $CR_j(t_j) \geq 0$ . In both cases, first-order conditions yield the critical points  $T_i/2$  and  $T_j/2$ , which are indeed maximum points. The custom revenue increases with the tariff  $t_i \in [0, T_i/2]$ , and it decreases with the tariff  $t_i \in [T_i/2, T_i]$ ,

$$0 = CR_i(0) = CR_i(T_i) \leq CR_i(t_i) \leq CR_i\left(\frac{T_i}{2}\right) = \frac{T_i^2}{6},$$

and so

$$t_i^{BR}(t_j, CR) = \frac{T_i}{2}, \quad t_j^{BR}(t_i, CR) = \frac{T_j}{2} \quad \text{and} \quad (t_i^N(CR), t_j^N(CR)) = \left(\frac{T_i}{2}, \frac{T_j}{2}\right).$$

The social utility  $CR_T(t_i, t_j)$  is

$$CR_T(t_i, t_j) = \frac{2t_i(T_i - t_i)}{3} + \frac{2t_j(T_j - t_j)}{3}$$

and so

$$t_i^{SR}(t_j; CR) = \frac{T_i}{2}, \quad t_j^{SR}(t_i; CR) = \frac{T_j}{2} \quad \text{and} \quad (t_i^S(CR), t_j^S(CR)) = \left(\frac{T_i}{2}, \frac{T_j}{2}\right).$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(CR) = t_i^N(CR) = \frac{T_i}{2}.$$

Therefore, the game with utility  $u_i = CR_i$ , is of the type **SE**.

**5.3. Consumers savings effects from the use of tariffs.** We now analyse the consumer's savings, i.e., the case where the utilities are given by the consumers surplus,  $u_i = CS_i$ .

The consumer surplus  $CS_i(t_i)$  of country  $X_i$  is

$$CS_i(t_i) \equiv CS_i(c_i, c_j; t_i) = \frac{(2(T_i + T_i^*) - t_i)^2}{18},$$

and the consumer surplus  $CS_j(t_j)$  of country  $X_j$  is

$$CS_j(t_j) \equiv CS_j(c_i, c_j; t_j) = \frac{(2(T_j + T_j^*) - t_j)^2}{18}.$$

The first-order conditions to minimize these quantities are

$$t_i = 2(T_i + T_i^*) \quad \text{and} \quad t_j = 2(T_j + T_j^*).$$

which are respectively bigger than  $T_i$  and  $T_j$ . So we have that

$$t_i^{BR}(t_j; CS) = 0, \quad t_j^{BR}(t_i; CS) = 0 \quad \text{and} \quad (t_i^N(CS), t_j^N(CS)) = (0, 0).$$

The social utility  $CS_T(t_i, t_j)$  is

$$CS_T(t_i, t_j) = \frac{(2(T_i + T_i^*) - t_i)^2}{18} + \frac{(2(T_j + T_j^*) - t_j)^2}{18}$$

and so

$$t_i^{SR}(t_j; CS) = 0, \quad t_j^{SR}(t_i; CS) = 0 \quad \text{and} \quad (t_i^S(CS), t_j^S(CS)) = (0, 0).$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(CS) = t_i^N(CS) = 0.$$

Therefore, the game with utility  $u_i = CS_i$ , is of the type **SE**.

**5.4. Firms profits effects from the use of tariffs.** We now consider the case where the utilities are the profits of the firms, i.e.,  $u_i = \pi_i$ .

The profit  $\pi_i(t_i, t_j)$  of the firm  $F_i$  is

$$\pi_i(t_i, t_j) \equiv \pi_i(c_i, c_j; t_i, t_j) = \frac{1}{9}[(2T_i^* + t_i)^2 + 4(T_j - t_j)^2].$$

and the profit of the firm  $F_j$  is

$$\pi_j(t_i, t_j) \equiv \pi_j(c_i, c_j; t_i, t_j) = \frac{1}{9}[(2T_j^* + t_j)^2 + 4(T_i - t_i)^2].$$

Thus, the profit  $\pi_i(t_i, t_j)$  increases with  $t_i$  and decreases with  $t_j$ , and vice-versa for the profit  $\pi_j(t_i, t_j)$ . So

$$t_i^{BR}(t_j, \pi) = T_i, \quad t_j^{BR}(t_i, \pi) = T_j \quad \text{and} \quad (t_i^N(\pi), t_j^N(\pi)) = (T_i, T_j).$$

The social utility  $\pi_T(t_i, t_j)$  is

$$\pi_T(t_i, t_j) = \frac{1}{9}[(2T_i^* + t_i)^2 + (2T_j^* + t_j)^2 + 4(T_i - t_i)^2 + 4(T_j - t_j)^2].$$

Hence,

$$\frac{\partial \pi_T}{\partial t_i} = \frac{4(T_i^* - 2T_i) + 10t_i}{9}.$$

Noting that

$$\frac{\partial^2 \pi_T}{\partial t_i^2} = \frac{10}{9} > 0,$$

we obtain that the local maxima of  $\pi_T$  is attained at the boundary points of the admissible tariffs

$$t_i^{SR}(t_j; \pi) \in \{0, T_i\}.$$

Similarly,

$$t_j^{SR}(t_i; \pi) \in \{0, T_j\}.$$

We have that

$$\pi_T(T_i, t_j) - \pi_T(0, t_j) = \frac{T_i}{9}(4T_i^* - 3T_i).$$

and

$$\pi_T(t_i, T_j) - \pi_T(t_i, 0) = \frac{T_j}{9}(4T_j^* - 3T_j).$$

Hence, a priori, there are four possibilities for the social optimum. However, the tariff pair  $(0, 0)$  cannot be achieved as a social optimum because the conditions  $4T_i^* < 3T_i$  and  $4T_j^* < 3T_j$  are incompatible. So we are left with three possible cases:

**Case I.**  $4T_i^* > 3T_i$  and  $4T_j^* > 3T_j$ . Equivalently,  $R_i > 3/4$  and  $R_j > 3/4$  or  $H_i > 3/5$  and  $H_j > 3/5$ . We have

$$t_i^{SR}(t_j; \pi) = T_i \quad \text{and} \quad t_j^{SR}(t_i; \pi) = T_j .$$

Thus,

$$(t_i^S(\pi), t_j^S(\pi)) = (T_i, T_j) .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(\pi) = t_i^N(\pi) = T_i .$$

Therefore, the game is of the type **SE**.

**Case II.**  $4T_i^* < 3T_i$ . Equivalently,  $R_i < 3/4$  or  $H_i < 3/5$ . The case  $4T_j^* < 3T_j$ , or equivalently,  $R_j > 3/4$  or  $H_j > 3/5$  is similar. We have

$$t_i^{SR}(t_j; \pi) = 0 \quad \text{and} \quad t_j^{SR}(t_i; \pi) = T_j .$$

Therefore,

$$(t_i^S(\pi), t_j^S(\pi)) = (0, T_j) .$$

Hence, there is a unique social optimum but it does not coincide with the Nash equilibrium

$$t_i^N(\pi) \neq t_i^S(\pi) \quad \text{and} \quad t_j^N(\pi) = t_j^S(\pi) = T_j .$$

Furthermore,

$$\pi_i(t_i^N, t_j^N) > \pi_i(t_i^S, t_j^S) \quad \text{and} \quad \pi_j(t_i^N, t_j^N) < \pi_j(t_i^S, t_j^S) .$$

Therefore, the game is of the type **LW**. More precisely, the outcome is  $\mathbf{L}_i \mathbf{W}_j$ .

**Case III.**  $4T_i^* = 3T_i$ . or equivalently  $R_i = 3/4$  or  $H_i = 3/5$ . The case  $4T_j^* = 3T_j$  or equivalently,  $R_j = 3/4$  or  $H_j = 3/5$  is similar. In this case

$$\pi_T(T_i, t_j) = \pi_T(0, t_j) .$$

So that

$$t_i^{SR}(t_j; \pi) = \{0, T_i\} \quad \text{and} \quad t_j^{SR}(t_i; \pi) = T_j .$$

Therefore, there are two social optima.

$$(t_i^S(\pi), t_j^S(\pi)) = (0, T_j) \quad \text{and} \quad (t_i^S(\pi), t_j^S(\pi)) = (T_i, T_j) .$$

One of them  $(t_i^S(\pi), t_j^S(\pi)) = (T_i, T_j)$  coincides with the Nash equilibrium, in which case the game is of **SE** type.

In the other social optimum,  $(t_i^S(\pi), t_j^S(\pi)) = (0, T_j)$ , by definition of Nash equilibrium we have that

$$\pi_i(t_i^N, t_j^N) > \pi_i(t_i^S, t_j^S) .$$

So, also by definition of social optimum we have

$$\pi_j(t_i^N, t_j^N) < \pi_j(t_i^S, t_j^S) ,$$

and hence the game is of **LW** type. More precisely, the outcome is  $\mathbf{L}_i \mathbf{W}_j$ .

5.5. **Game outcomes.** In this section we further discuss the game outcomes obtained for different utilities given by relevant economic quantities.

For every pair of tariffs  $(t_i, t_j)$ , we found the Nash equilibrium for the second sub-game, i.e. the home and export quantities such that firms competitively maximize their profits. Then, using the Nash equilibrium for the home and export quantities we found the tariffs that lead to a Nash equilibria or to a social equilibria for different utilities.

We observed that for the home quantities the Nash equilibria and the social optimal tariffs are the same and equal to the maximal tariffs. So, countries decide to block exports, both when in competition and in cooperation. For the export quantities all tariffs lead to a Nash equilibrium but only the  $(0, 0)$  tariffs is a social optimum, with all the others yielding a prisoner’s dilemma game. For the aggregate quantities in the market of each country, prices, custom revenues and consumer surpluses we found that the Nash tariffs coincide with the social tariffs, thus, the games with these utilities are of Social Equilibrium (**SE**) type. For the aggregate quantities and the consumer surpluses the tariffs are zero, corresponding to free export; for the custom revenues they are half of the maximal tariffs; and for prices they are the maximal tariffs. Hence, some of the difficulties of imposing tariffs arise from these social equilibria having different tariffs. We summarize these results in Table 1.

	<b>SE game</b>					
Economic quantity	$h$	$e$	$Q$	$p$	$CR$	$CS$
Nash (Social) tariff of country i	$T_i$	0	0	$T_i$	$T_i/2$	0
Nash (Social) tariff of country j	$T_j$	0	0	$T_j$	$T_j/2$	0

Table 1: The Nash (Social) tariffs for the home quantities, total quantity in the market, inverse demand, custom revenue and custom surplus, resulting in a social equilibrium.  $h$  - Home quantities;  $Q$  - Aggregate quantity in each country;  $p$  - Inverse demand;  $CR$  - Custom revenue;  $CS$  - Consumer surplus.

For the total quantities produced by the firms we found that the Nash tariffs are the maximal tariffs, and the social tariffs are the zero tariffs. The game can be either of Prisoner’s Dilemma (**PD**) type or of Lose-Win (**LW**) type, depending on the maximal tariffs. When the maximal tariffs of the two countries are sufficiently similar, then the game is of **PD** type. The game is **LW** for the bounds presented in Table 2.

Total quantities $(q_i, q_j)$ produced by the firms			
Condition	Nash tariffs	Social tariffs	Game type
If $2T_j < T_i$	$(T_i, T_j)$	$(0, 0)$	<b>L<sub>i</sub>W<sub>j</sub></b>
If $T_i/2 \leq T_j \leq 2T_i$	$(T_i, T_j)$	$(0, 0)$	<b>PD</b>
If $2T_i < T_j$	$(T_i, T_j)$	$(0, 0)$	<b>L<sub>j</sub>W<sub>i</sub></b>

Table 2: Comparing total quantities of the two countries with Nash tariffs and social tariffs with different cost similarities and concluding the game type.

For the profits of the firms we found that the Nash tariffs are the maximal tariffs and the social tariffs can be either the maximal tariffs, or one of the two symmetrical cases where one country chooses the maximal tariff and the other chooses the zero tariff. Which one of the cases occurs depends on the tax-free home

production indexes. When a firm has a tax-free home production index lower than the threshold  $3/5$ , the game is of **LW** type, and this firm has a profit loss. In this cases, the country with the higher tax-free home production index earns more profit at the social optimum. The social optimum tariff for that country with the tax-free home production index below the threshold is to become tax-free, and he has a loss is profit. Since that country has a lower tax-free home production index, this means he has greater loss while becoming tax-free, meaning that the firm is less competitive when his home market is shared between the two countries. Hence, its firm produces its tax-free home production quantity and does not export, while the firm of the other country has the monopoly in its own market and exports for the tax-free country. So, the less competitive firm has interest in becoming tax-free if he gets some compensation by the other firm. Possibilities that may be discussed between the two countries might include merging of the two firms, R&D exchange, or allowing representation of the more competitive firm to be settled in the less competitive country. When both firms have a high tax-free home production index the game is of **SE** type. In this case both firms produce their monopoly quantities. We summarize these results in Table 3.

Profits $(\pi_i, \pi_j)$ of the firms			
Condition	Nash tariffs	Social tariffs	Game type
If $H_i < 3/5$	$(T_i, T_j)$	$(0, T_j)$	<b>L<sub>i</sub>W<sub>j</sub></b>
If $H_i > 3/5$ and $H_j > 3/5$	$(T_i, T_j)$	$(T_i, T_j)$	<b>SE</b>
If $H_j < 3/5$	$(T_i, T_j)$	$(T_i, 0)$	<b>L<sub>j</sub>W<sub>i</sub></b>

Table 3: Comparing profits of the firms of the two countries with Nash tariffs and social tariffs, where  $H_i$  and  $H_j$  are the tax-free home production indexes.

**6. Nash and social welfares.** In this section, we consider the utility of the governments to be the welfare of the country, i.e.  $u_i = W_i$ . We will compute the Nash equilibrium tariffs and the social tariffs and analyse the game type obtained according to the tax-free home production index.

**6.1. Computation of the equilibria.** The welfare  $W_i(t_i, t_j)$  of the country  $X_i$  is given by

$$W_i(t_i, t_j) = \frac{1}{9} \left[ (2T_i^* + t_i)^2 + 4(T_j - t_j)^2 \right] + \frac{2}{3} t_i (T_i - t_i) + \frac{1}{18} (2(T_i + T_i^*) - t_i)^2 .$$

The welfare  $W_j(t_i, t_j)$  of the country  $X_j$  is given by

$$W_j(t_i, t_j) = \frac{1}{9} \left[ (2T_j^* + t_j)^2 + 4(T_i - t_i)^2 \right] + \frac{2}{3} t_j (T_j - t_j) + \frac{1}{18} (2(T_j + T_j^*) - t_j)^2 .$$

We have that

$$\frac{\partial W_i}{\partial t_i} = \frac{4T_i + 2T_i^*}{9} - t_i , \quad \frac{\partial W_j}{\partial t_j} = \frac{4T_j + 2T_j^*}{9} - t_j ,$$

and

$$\frac{\partial^2 W_i}{\partial t_i^2} = -1 , \quad \frac{\partial^2 W_j}{\partial t_j^2} = -1 .$$

Therefore, the maximum points of the polynomials  $W_i(t_i, t_j)$  and  $W_j(t_i, t_j)$  in  $t_i$  and  $t_j$  are, respectively

$$A_{W,i} = \frac{2(T_i^* + 2T_i)}{9} > 0 \quad \text{and} \quad A_{W,j} = \frac{2(T_j^* + 2T_j)}{9} > 0 .$$

Noting that  $A_{W,i} < T_i$  is equivalent to  $2T_i^* < 5T_i$ , and that  $A_{W,j} < T_j$  is equivalent to  $2T_j^* < 5T_j$ , we get that the best responses are

$$t_i^{BR}(t_j, W) = \begin{cases} A_{W,i} , & \text{if } T_i^* < \frac{5T_i}{2} ; \\ T_i , & \text{otherwise .} \end{cases} \quad t_j^{BR}(t_i, W) = \begin{cases} A_{W,j} , & \text{if } T_j^* < \frac{5T_j}{2} ; \\ T_j , & \text{otherwise .} \end{cases}$$

The social utility  $W_T(t_i, t_j)$  is

$$W_T(t_i, t_j) = W_i(t_i, t_j) + W_j(t_i, t_j) .$$

Hence, we have that

$$\frac{\partial W_T}{\partial t_i} = \frac{2T_i^* - 4T_i - t_i}{9} .$$

and

$$\frac{\partial^2 W_T}{\partial t_i^2} = -\frac{1}{9} .$$

Let

$$B_{W_S,i} = 2(T_i^* - 2T_i) ,$$

and analogously,

$$B_{W_S,j} = 2(T_j^* - 2T_j) .$$

Noting that  $0 < B_{W_S,i} < T_i$  is equivalent to  $2T_i < T_i^* < 5T_i/2$ , we get that the social best responses are

$$t_i^{SR}(t_j; W) = \begin{cases} 0 , & \text{if } T_i^* \leq 2T_i ; \\ B_{W_S,i} , & \text{if } 2T_i < T_i^* < \frac{5T_i}{2} ; \\ T_i , & \text{if } T_i^* \geq \frac{5T_i}{2} . \end{cases}$$

Similarly, we get

$$t_j^{SR}(t_i; W) = \begin{cases} 0 , & \text{if } T_j^* \leq 2T_j ; \\ B_{W_S,j} , & \text{if } 2T_j < T_j^* < \frac{5T_j}{2} ; \\ T_j , & \text{if } T_j^* \geq \frac{5T_j}{2} . \end{cases}$$

Hence, there are several possible cases. Since  $0 \leq H_i \leq 2/3 \leq H_j \leq 1$  or  $0 \leq H_j \leq 2/3 \leq H_i \leq 1$  some possibilities are incompatible with each other. So we are left with five cases:

**Case I.**  $T_i^* \leq 2T_i$  and  $T_j^* \leq 2T_j$ . Equivalently,  $R_i \leq 2$  and  $R_j \leq 2$  or  $H_i \leq 4/5$  and  $H_j \leq 4/5$ . The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (A_{W,i}, A_{W,j}) .$$

The social optimum is

$$(t_i^S(W), t_j^S(W)) = (0, 0) .$$

The social optimum does not coincide with the Nash equilibrium.

The welfare at the Nash equilibrium is

$$W_j(A_{W,i}, A_{W,j}) = CR_j(A_{W,j}) + CS_j(A_{W,j}) + \pi_j(A_{W,i}, A_{W,j}) ,$$

where

$$CR_j(A_{W,j}) = -\frac{4(T_j^* + 2T_j)(2T_j^* - 5T_j)}{243} ,$$

$$CS_j(A_{W,j}) = \frac{2(8T_j^* + 7T_j)^2}{729}$$

and

$$\pi_j(A_{W,i}, A_{W,j}) = \frac{4(4(5T_j^* + T_j)^2 + (2T_i^* - 5T_i)^2)}{729} .$$

The welfare at the social optimum is

$$W_j(0, 0) = CS_j(0) + \pi_j(0, 0) ,$$

where

$$CS_j(0) = \frac{2(T_j^* + T_j)^2}{9}$$

and

$$\pi_j(0, 0) = \frac{4}{9} ((T_j^*)^2 + T_j^2) .$$

Letting

$$\Delta W_{1,j} = W_j(A_{W,i}, A_{W,j}) - W_j(0, 0) ,$$

we have

$$\Delta W_{1,j} = \frac{2(9(T_j^* + 2T_j)^2 - 8(2T_i + T_i^*)(7T_i - T_i^*))}{729} .$$

**Case Ia).** When  $R_i \neq 1$ , and so  $R_j \neq 1$ ,

$$\Delta W_{1,j} = \frac{2}{729} \left( \frac{9(R_i - 1)^2(R_j + 2)^2}{(R_j - 1)^2} + 8(R_i + 2)(R_i - 7) \right) .$$

Hence, depending on the ratios  $R_i$  and  $R_j$ , or the tax-free home production indexed  $H_i$  and  $H_j$ , the game has three outcomes (see figure 1): (1) If  $R_i = 1 - \beta$  and  $R_j = 1 + \beta$ , with  $\beta$  close to 0, the game is of Prisoner's dilemma (**PD**) type; (2) If  $R_i$  is closer to 1 than  $R_j$ , the outcome is (**L<sub>i</sub>W<sub>j</sub>**); (3) If  $R_j$  is closer to 1 than  $R_i$  the outcome is (**L<sub>j</sub>W<sub>i</sub>**).

**Case Ib).** If  $R_i = 1$  (equivalently  $R_j = 1$ ), or  $H_i = 2/3$  and  $H_j = 2/3$ , we obtain

$$\begin{aligned} \Delta W_{1,i} &= \frac{2(9T_i^2 - 16T_j^2)}{81} , \\ \Delta W_{1,j} &= \frac{2(9T_j^2 - 16T_i^2)}{81} . \end{aligned}$$

So, we have that

$$\Delta W_{1,i} > 0 \text{ iff } 3/4T_i > T_j ,$$

$$\Delta W_{1,j} > 0 \text{ iff } T_j > 4/3T_i ,$$

Hence, depending on  $T_i$  and  $T_j$ , the game has three outcomes: (1) If  $T_i$  is close to  $T_j$ , the game is of Prisoner's dilemma (**PD**) type; (2) when  $T_i$  is sufficiently larger than  $T_j$ , the outcome is  $(\mathbf{L}_i \mathbf{W}_j)$ ; (3) when  $T_j$  is sufficiently larger than  $T_i$ , the outcome is  $(\mathbf{L}_j \mathbf{W}_i)$ .

**Case II.**  $2T_j < T_j^* < 5T_j/2$ . Equivalently,  $2 < R_j < 5/2$  or  $4/5 < H_j < 5/6$ . In this case we also have that  $T_i^* \leq 2T_i$ , equivalently,  $R_i \leq 2$  or  $H_i \leq 4/5$ . The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (A_{W,i}, A_{W,j}) .$$

Hence, the welfare at the Nash equilibrium is the same as in case ia). The social optimum is

$$(t_i^S(W), t_j^S(W)) = (0, B_{W_S,j}) .$$

Since

$$B_{W_S,j} < A_{W,j}$$

is equivalent to

$$T_j^* < 5/2T_j$$

which is verified in this case, then

$$t_i^S(W) \neq t_i^N(W) \quad \text{and} \quad t_j^S(W) \neq t_j^N(W) .$$

The welfare at the social optimum is

$$W_j(0, B_{W_S,j}) = CR_j(B_{W_S,j}) + CS_j(B_{W_S,j}) + \pi_j(0, B_{W_S,j}) ,$$

where

$$CR_j(B_{W_S,j}) = -\frac{4(T_j^* - 2T_j)(2T_j^* - 5T_j)}{3} ,$$

$$CS_j(B_{W_S,j}) = 2T_j^2$$

and

$$\pi_j(0, B_{W_S,j}) = \frac{4(4(T_j^* - T_j)^2 + T_i^2)}{9} .$$

Letting

$$\Delta W_{2,i} = W_i(A_{W,i}, A_{W,j}) - W_i(0, B_{W_S,j})$$

and

$$\Delta W_{2,j} = W_j(A_{W,i}, A_{W,j}) - W_j(0, B_{W_S,j}) .$$

We have

$$\begin{aligned} \Delta W_{2,i} &= \frac{2(9(2T_i + T_i^*)^2 - 160(2T_j^* - 5T_j)^2)}{729} \\ \Delta W_{2,j} &= \frac{16(18(2T_j^* - 5T_j)^2 - (2T_i + T_i^*)(7T_i - T_i^*))}{729} . \end{aligned}$$

Hence,

$$\begin{aligned} \Delta W_{2,i} &= \frac{2}{729} \left( \frac{9(R_j - 1)^2(R_i + 2)^2}{(R_i - 1)^2} - 160(2R_j - 5)^2 \right) \\ \Delta W_{2,j} &= \frac{16}{729} \left( \frac{18(R_i - 1)^2(2R_j - 5)^2}{(R_j - 1)^2} - (R_i + 2)(7 - R_i) \right) . \end{aligned}$$

Thus, depending on the ratios  $R_i$  and  $R_j$ , the game has three outcomes (see figure 1): (i) For instance, if  $R_i$  is close to 0 and  $R_j$  close to 2 the outcome is  $\mathbf{L}_j \mathbf{W}_i$ ; (ii) For instance, if  $R_i$  is close to 0, but not too close, and  $R_j$  is close to 2, but not too close, the game is of Prisoner's dilemma (**PD**) type; (iii) For the majority of values of  $R_i$  and  $R_j$  the outcome is  $\mathbf{L}_i \mathbf{W}_j$ .

**Case III.**  $0 < 5T_j/2 \leq T_j^*$ . Equivalently,  $R_j > 5/2$  or  $5/6 < H_j < 1$ . In this case we also have that  $T_i^* \leq 2T_i$ , equivalently,  $R_i \leq 2$  or  $H_i \leq 4/5$ . The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (A_{W,i}, T_j) .$$

The social optimum is

$$(t_i^S(W), t_j^S(W)) = (0, T_j) .$$

The Welfare at the Nash equilibrium is

$$W_j(A_{W,i}, T_j) = CS_j(T_j) + \pi_j(A_{W,i}, T_j) ,$$

where

$$CS_j(T_j) = \frac{(2T_j^* + T_j)^2}{18}$$

and

$$\pi_j(A_{W,i}, T_j) = \frac{81(2T_j^* + T_j)^2 + 4(2T_i^* - 5T_i)^2}{729} .$$

The welfare at the social optimum is

$$W_j(0, T_j) = CS_j(T_j) + \pi_j(0, T_j) ,$$

where

$$\pi_j(0, T_j) = \frac{(2T_j^* + T_j)^2 + 4T_i^2}{9} .$$

Furthermore, by definition of Nash equilibrium, clearly we have that

$$W_i(A_{W,i}, T_j) > W_i(0, T_j) .$$

The fact that the social optimum is  $(0, T_j)$  together with this last inequality yields

$$W_j(A_{W,i}, T_j) < W_j(0, T_j) .$$

Hence, in this case there is a unique social optimum that does not coincide with the Nash equilibrium. Furthermore, the game is of the type **LW**. More precisely, the outcome is  $\mathbf{L}_i \mathbf{W}_j$ .

**Case IV.**  $2T_i < T_i^* < 5T_i/2$ . This case is dual to case II.

**Case V.**  $0 < 5T_i/2 \leq T_i^*$ . This case is dual to case III.

6.2. **Welfare game outcomes.** Here, we present the regions where the outcomes are of type  $\mathbf{L}_i\mathbf{W}_j$ ,  $\mathbf{PD}$  and  $\mathbf{L}_j\mathbf{W}_i$ . We will analyse the case  $0 < H_i < 2/3 < H_j < 1$ , the other case is dual to this one. Observe that in the case we study, since  $H_i$  is lower than  $H_j$ , country  $X_i$  has a higher decrease in home quantities when changing from maximal tariffs to a tax-free situation. The corner point  $H_i = 2/3$  and  $H_j = 2/3$  was analysed separately before (case Ib)).

For the welfare of the countries we found two thresholds for the tax-free home production index  $H_j$ : the *social monopoly-tax threshold*  $5/6$  and the *social tax-free threshold*  $4/5$  (see figure 1). For all values of the tax-free home production indexes  $H_i$  and  $H_j$ , the firm  $F_i$  chooses the Nash tariff  $A_{W,i}$  and vanishes its tariff at the social equilibrium. In case III, when  $H_j \geq 5/6$ , the game is of  $\mathbf{L}_i\mathbf{W}_j$  type and the country  $X_j$  has a welfare gain. The country  $X_j$  applies the maximal tariff both at the Nash and Social equilibria. When  $H_j < 5/6$ , the game has three outcomes: prisoner’s dilemma  $\mathbf{PD}$ , and lose-win  $\mathbf{L}_i\mathbf{W}_j$  and  $\mathbf{L}_j\mathbf{W}_i$ . In this case, firm  $F_j$  chooses the Nash tariff  $A_{W,j}$ . Its social tariff is  $B_{W_s,j}$  if  $4/5 < H_j < 5/6$ , and it vanishes its tariff at the social equilibrium if  $2/3 < H_j \leq 4/5$  (see table 4). We observe from figure 1 that in case II the majority of the parameter yields either a  $\mathbf{PD}$  or a  $\mathbf{L}_i\mathbf{W}_j$  type game, meaning that country  $X_j$  has a welfare gain, except for a very small parameter region where country  $X_j$  has a welfare loss. In case I, the three game types may occur. When  $H_j$  gets lower and closer to  $2/3$  (meaning that the tax-free home production of  $X_j$  gets lower in comparison to the monopoly home quantity), it gets more likely that the game type is  $\mathbf{L}_j\mathbf{W}_i$ , with country  $X_j$  losing welfare. For lower values of  $H_i$ ,  $H_j$  doesn’t need to be so lower in order to have a  $\mathbf{L}_j\mathbf{W}_i$  type game, and there is a threshold in  $H_i$  (approximately 0.2) such that the game is always of this type in case I.

Welfares ( $W_i, W_j$ ) of the countries			
Condition	Nash tariffs	Social tariffs	Game type
$H_j \geq 5/6$	$(A_{W,i}, T_j)$	$(0, T_j)$	$\mathbf{L}_i\mathbf{W}_j$
$4/5 < H_j < 5/6$	$(A_{W,i}, A_{W,j})$	$(0, B_{W_s,j})$	$\mathbf{LW}$ or $\mathbf{PD}$
$H_j \leq 4/5$	$(A_{W,i}, A_{W,j})$	$(0, 0)$	$\mathbf{LW}$ or $\mathbf{PD}$

Table 4: Comparing welfares of the two countries with Nash tariffs and social tariffs where  $H_i$  and  $H_j$  are the tax-free home production indexes satisfying  $0 < H_i < 2/3 < H_j < 1$ .

We have observed that  $B_{W_s,j} < A_{W,j}$ . So, we have that for both countries the social tariffs are lower than the Nash tariffs, and they are lower or both countries except for the situation where one of the countries plays its maximal tariff. So, any trade agreement that enforces the social tariffs will therefore yield lower tariffs than the tariffs used at a competitive (Nash) equilibrium for at least one of the countries. By direct inspection of the custom revenue, consumer surplus and profit functions, we find that if the two countries decide to impose the social tariffs the following holds: a) the custom revenue of country  $X_i$  vanishes, since his tariff vanishes at the social optimum, and in some situations (cases I and III), it also vanishes for country  $X_j$ , since country  $X_j$ ’s social tariff is, respectively, 0 and the maximal tariff. When it doesn’t vanish, which occurs only for values of  $H_j$  between the two thresholds, then the its custom revenue may increase or decrease; b) the consumer surplus of the countries always increases; and c) the profits may increase or decrease.

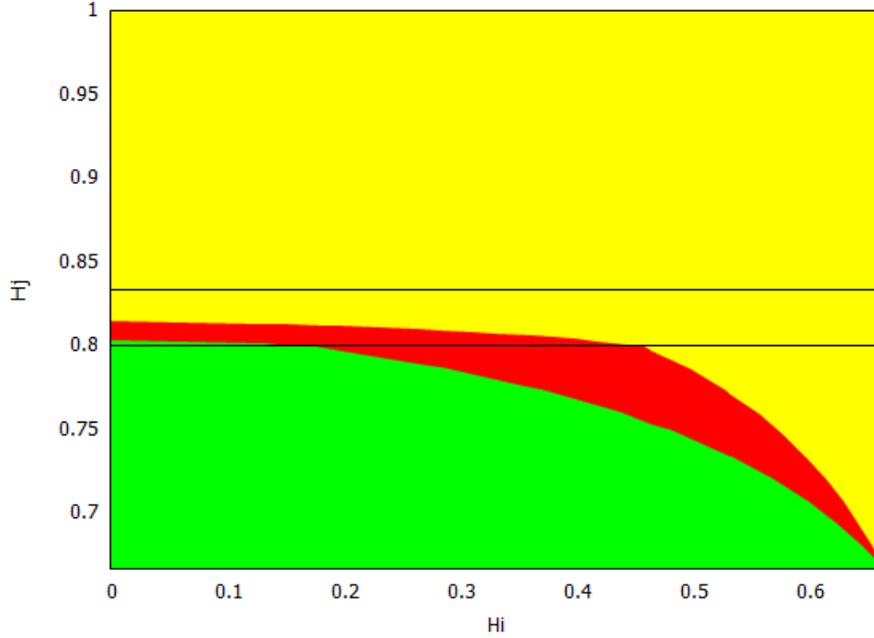


FIGURE 1. The Welfare Game Type: Green -  $L_j W_i$ ; Red - PD; Yellow -  $L_i W_j$ .

**7. Conclusions.** We have considered an international trade model with two countries, where the governments of each country choose whether or not to impose tariffs in the import. We have considered a two-stage game, where in the first stage, governments choose their tariffs and in the second stage, firms in each country competitively choose their home and export quantities. For every pair of tariffs, we found the Nash equilibria for the second sub-game. For the first sub-game the governments can choose competitive (Nash) tariffs or social (cooperative) tariffs. We considered different economic quantities as the utility of the governments, namely, total quantities in the home market, total quantities produced by the firms, prices, profits of the firms, consumer's savings, custom revenues of the countries and the welfare of the country. For each utility, we have classified the game according to the social and Nash tariffs of the governments. For the welfare of the countries we proved that the outcome of the game is either a prisoner's dilemma or a lose-win dilemma. This classification for the different utilities and for the welfare suggests where the difficulties in establishing trade agreements may appear, since in a lose-win dilemma the losing country has to be somehow compensated by means of a trade agreement in order to accept the enforcement of the social tariffs. Furthermore, a trade agreement might present some externalities that need to be considered.

Future work can consist, for instance, in introducing a trade agreement in our current setting, and study the parameter regions where difficulties and externalities may occur, to study conditions for the enforcement of the agreement, ideally rendering it durable in time, to introduce and study the effects of the swap of R&D between the two firms to decrease their production costs, or to include the effects of subsidies, fines, price dumping and merging and shut-down of firms. The inclusion

of one or more of these features into the trade agreement may be a way to overcome some of the externality effects that may arise.

**Acknowledgments.** The authors would like to thank Francis Bloch and the two anonymous referees for their suggestions. This work is financed by the ERDF - European Regional Development Fund through the Operational Programme for Competitiveness and Internationalisation - COMPETE 2020 Programme within project “POCI-01-0145-FEDER-006961”, and by National Funds through the FCT - Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) as part of project UID/EEA/50014/2013, and within project “Dynamics, Optimization and Modelling”, with reference PTDC/MAT-NAN/6890/2014. Part of this work was done during visits of the authors to Instituto de Matemática Pura e Aplicada - IMPA, Brasil, to whom they thank their hospitality. Alberto Adrego Pinto thanks LIAAD-INESC TEC and the financial support received through the Special Visiting Researcher Program (Bolsa Pesquisador Visitante Especial - PVE) “Dynamics, Games and Applications”, with reference 401068 / 2014-5 (call: MEC / MCTI / CAPES / CNPQ / FAPS) at IMPA, Rio de Janeiro, Brasil. F. Martins thanks the financial support of FCT - Fundação para a Ciência e a Tecnologia - through a PhD. grant of the MAP-PDMA program with reference PD / BD / 105726 / 2014 and the financial support of Instituto de Matemática Pura e Aplicada - IMPA, at the occasion of the 16<sup>th</sup> SAET Conference on Current Trends in Economics held in Rio de Janeiro, Brasil, where this work was presented. J. P. Zubelli was supported by CNPq grants 302161/2003-1 and 474085/2003-1 and by FAPERJ through the programs ‘Cientistas do Nosso Estado’ and ‘Pensa Rio’.

#### REFERENCES

- [1] K. Bagwell and R. Staiger, A theory of managed trade, *American Economic Review*, **80** (1990), 779–795.
- [2] K. Bagwell and R. Staiger, Enforcement, private political pressure, and the GAT/WTO escape clause, *Journal of Legal Studies*, **34** (2005), 471–513.
- [3] N. Banik, F. A. Ferreira, J. Martins and A. A. Pinto, *An Economical Model for Dumping by Dumping in a Cournot Model*, chapter in “Dynamics, Games and Science II: DYNA 2008, in Honour of Maurício Peixoto and David Rand”, editors: M. M. Peixoto, A. A. Pinto and D. A. Rand, pp. 141–154, Springer, 2011.
- [4] J. A. Brander, Intra-industry trade in identical commodities, *Journal of International Economics*, **11** (1981), 1–14.
- [5] J. A. Brander and B. J. Spencer, Export subsidies and international market share rivalry, *Journal of International Economics*, **18** (1985), 83–100.
- [6] J. I. Bulow, J. D. Geanakoplos and P. D. Klemperer, Multi-market oligopoly: Strategic substitutes and complements, *Journal of Political Economy*, **93** (1985), 488–511.
- [7] M. Choubdar, J. P. Zubelli and A. A. Pinto, Nash and Social Welfare Impact in International Trade, Recent Advances in Applied Economics, *Proceedings of the 6th International Conference on Applied Economics, Business and Development (AEBD’14) Lisbon, Portugal*, (2014), 23–26.
- [8] M. Choubdar, E. Farias, F. A. Ferreira and A. A. Pinto, Uncertainty costs on an international duopoly with tariffs, *Proceedings of the 6th International Conference on Applied Economics, Business and Development (AEBD’14) Lisbon, Portugal*, (2014), 13–16.
- [9] A. Dixit, International trade policy for oligopolistic industries, *Economic Journal*, supplement, **94** (1984), 1–16.
- [10] A. Dixit, Strategic aspects of trade policy, in *Bewley, T. (Editor), Advances in Economic Theory*, Cambridge University Press, 329–362.
- [11] A. Dixit and G. Grossman, Targeted export promotion with several oligopolistic industries, *Journal of International Economics*, **21** (1986), 233–249.

- [12] G. Eaton and G. Grossman, [Optimal trade and industrial policy under oligopoly](#), *Quarterly Journal of Economics*, **101** (1984), 383–406.
- [13] F. A. Ferreira, *Applications of Mathematics to Industrial Organization*, Ph.D. Thesis, Universidade do Porto, Portugal, 2007.
- [14] F. A. Ferreira, F. Ferreira, M. Ferreira and A. A. Pinto, Quantity competition in a differentiated duopoly, *Chapter in J. A. Tenreiro Machado, Bela Patkai and Imre J. Rudas (Eds.): Intelligent Engineering Systems and Computational Cybernetics. Springer Science+Business Media B.V.*, (2008), 365–374.
- [15] F. A. Ferreira, F. Ferreira and A. A. Pinto, Flexibility in Stackelberg leadership, *Chapter in J. A. Tenreiro Machado, Bela Patkai and Imre J. Rudas (Eds.): Intelligent Engineering Systems and Computational Cybernetics, Springer Science+Business Media B.V.*, (2008), 399–405.
- [16] F. A. Ferreira, F. Ferreira and A. A. Pinto, Bayesian price leadership, *Chapter in Tas, K. et al. (eds.): Mathematical Methods in Engineering*, Springer, Dordrecht, (2007), 371–379.
- [17] F. A. Ferreira, F. Ferreira and A. A. Pinto, Unknown costs in a duopoly with differentiated products, *Chapter in Tas, K. et al. (eds.): Mathematical Methods in Engineering*, Springer, Dordrecht, (2007), 359–369.
- [18] F. A. Ferreira, F. Ferreira and A. A. Pinto, Uncertainty on an asymmetric duopoly, Progress in Industrial Mathematics at ECMI 2006, Proceedings 14th European Conference on Mathematics for Industry, Madrid, Spain, July 10-14. Springer, Berlin (2006) to appear.
- [19] E. O’N. Fisher and C. A. Wilson, Price competition between two international firms facing tariffs, *International Journal of Industrial Organization*, **13** (1995), 67–87.
- [20] R. Gibbons, *A Primer in Game Theory*, Pearson Prentice Hall, Harlow, 1992.
- [21] G. M. Grossman, Strategic export promotion: A critique, *Chapter 3 in: Strategic Trade Policy and the New International Economics*, P. R. Krugman (editor), MIT Press, Cambridge MA, (1986), 47–68.
- [22] G. V. Haberler, *The Theory of International Trade with its Application to Commercial Policy*, Macmillan, New York, 1937.
- [23] E. Helpman, *Increasing Returns, Imperfect Markets, and Trade Theory*, Jones, R.W., Kenen, P.B. (eds.): Handbook of International Economics, **1** North Holland Press, Amsterdam, 1984, Chapter 7.
- [24] J.-M. M. Kilolo, *Country Size, Trade Liberalization and Transfers*, MPRA Paper, University Library of Munich, Germany, 2013.
- [25] K. Krishna, Trade restrictions as facilitating practices, *Journal of International Economics*, **26** (1989), 251–270.
- [26] P.-C. Liao, Rivalry between exporting countries and an importing country under incomplete information, *Academia Economic Papers*, **32** (1990), 605–630.
- [27] N. Lim ‘ao and K. Saggi, Tariff retaliation versus financial compensation in the enforcement of international trade agreements, *Journal of International Economics*, **76** (2008), 48–60.
- [28] N. Lim ‘ao and K. Saggi, Size inequality, coordination externalities and international trade agreements, *European Economic Review*, **63** (2013), 10–27.
- [29] J. Martins, N. Banik and A. A. Pinto, [A Repeated Strategy for Dumping](#), to appear in Discrete Dynamical Systems and Applications, ICDEA 2012, editors: Lluís Alseda, Jim M. Cushing, Saber Elaydi and A. A. Pinto, Springer.
- [30] J. Martins and A. A. Pinto, Deviation from collusion with and without dumping, to appear in Modeling, Dynamics, Optimization and Bioeconomics II, editors: A. A. Pinto and D. Zilberman, Springer.
- [31] J. McMillan, *Game Theory in International Economics*, Harwood Academic Publishers, Chur, Switzerland, 1986.
- [32] A. A. Pinto, B. M. Oliveira, F. A. Ferreira and F. Ferreira, Stochasticity favoring the effects of the R&D strategies of the firms, *Chapter in J. A. Tenreiro Machado et al. (Eds.): Intelligent Engineering Systems and Computational Cybernetics*, Springer Science+Business Media B.V., (2009), 415–423.
- [33] A. A. Pinto, B. M. Oliveira, F. A. Ferreira and F. Ferreira, Investing to survive in a duopoly model, *Chapter in J. A. Tenreiro Machado et al. (Eds.): Intelligent Engineering Systems and Computational Cybernetics*, Springer Science+Business Media B.V., (2009), 407–414.
- [34] B. J. Spencer and J. A. Brander, [International R&D rivalry and industrial strategy](#), *Review of Economic Studies*, **50** (1983), 707–722.

- [35] R. Staiger, International rules and institutions for trade policy, in: *Grossman, Gene, Rogoff, Kenneth (Eds.): Handbook of international economics*, vol. 3, Elsevier, North-Holland, 1495–1551.

Received December 2015; revised January 2017.

*E-mail address:* [philip\\_m90@hotmail.com](mailto:philip_m90@hotmail.com)

*E-mail address:* [aapinto1@gmail.com](mailto:aapinto1@gmail.com)

*E-mail address:* [zubelli@impa.br](mailto:zubelli@impa.br)