Optimal positioning of autonomous marine vehicles for underwater acoustic source localization using TOA measurements

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***Abstract*—In opposition to the surface, no common solution is available for localization of active objects underwater. Typical solutions use acoustics as a means to implicitly measure ranges or angles and consequently determine the position of a transmitter. If the receivers are synchronized among themselves, the position of the transmitter can be estimated based on the time-of-arrivals (TOA). The confidence on the estimate varies with respect to the relative positions of the receivers and the transmitter. In this paper, we present recent developments for optimal 3D positioning of TOA sensors based on the a metric that uses the Fisher information matrix. We give the necessary conditions to obtain the best possible estimate. To our best knowledge, no analytical solution has been yet presented for this problem. We complete and validate our study with a simulation of optimal positioning of four TOA sensors.**

I. Introduction

Underwater localization and positioning is still a considerable challenge due to environmental and technological constraints. Such limitations have been restraining the growth of autonomous robotic system in such environments and therefore the localization of underwater targets, including mobile robots, has attracted the interest of several researchers over the last years [1]–[8]. In this paper, we tackle the optimal three-dimensional positioning of time-of-arrival (TOA) sensors to track an underwater sound source. Although motivated by underwater applications, the work presented here has applications in general networks of distributed sensors that make use of TOA measurements to estimate the position of a source.

Underwater localization solutions predominantly include two types of methods: Range-based and combined angle/range-based. Range based solutions use trilateration to determine the position of an active transmitter underwater. The so-called Long Baseline (LBL) has been employed in many georeferentiation applications for which bounded error localization is required [1]–[4]. LBL systems may require precise synchronization between of the transmitter and the receivers for one-way travel time (OWTT) [8]. In opposition, two-way travel time (TWTT) based systems do not need synchronization [2]. While the former can be composed by two set of sensors (active transmitters and passive receivers), the second scheme must use both in each beacon or use transducers that accumulate the two roles.

The Ultra-Short Baseline (USBL) systems (see, for example, [9]) use the bearing and ranging capabilities of acoustic receivers and transmitters. Receivers are placed closely measuring the time difference of arrival, thus making it possible to compute the angle of arrival. The range measurements are obtained in a similar fashion to the one of LBL.

Self localization can still be achieved by means of Inertial Navigation Systems (INS) composed by Inertial Measurement Units (IMU) to measure angles, angle rates and accelerations, and possibly a Doppler Velocity Log (DVL) to measure the linear velocity of the robotic platform. Note that localization using inertial measurements corrupted by biases does not ensure bounded error on the position estimate for unlimited time.

All these technique have been used in several self-localization problems, which are a fundamental part of autonomous vehicle navigation. In order to estimate its position, a vehicle measure its relative ranges or angles to one or more beacons, whose positions are known. Complementary, tracking techniques typically have to measure variables in different places in order to infer the position of the target object. This is the case of tracking using TOA sensors. Under some conditions, LBL and USBL systems can also be used for target tracking. However, the combined use of simple hydrophones and acoustic pingers remains considerably less expensive than LBL and USBL systems.

Optimal sensor positioning for tracking purposes has recently attracted the attention of several researchers [10]–[17]. Numerous algorithm have been developed for estimation. The most recurrent in the literature are the Kalman filter [18], the particle filter and least-squares methods. We refer to [19] for an overview on estimation applied to robotics. In several problems, the estimate variance depends on the state itself and some works have tackled the problem so that the state is guaranteed to be in the vicinity of the optimal observability points. The Cramer-Rao lower bound (CRLB), which provides a measure of the achievable performance of an efficient estimator, has been used in several works to assess such *measure of observability* (see, for example, [10], [11], [20]).

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Mostly based on the analysis of the Fisher information matrix (FIM), whose determinant is the inverse of the CRLB, several results have been developed for different scenarios, including two- and three-dimensional problems with homogeneous and heterogeneous sensors. Different types of sensors that provide measurements on the bearing, range, received signal strength, time of arrival, time difference of arrival have been considered. Significant contributions have been given in [10], [11], [14] and two common approaches are considered in the literature: Minimization of the average of the variances (A-optimality) or minimization of the volume of the confidence regions (D-optimality) (we refer to [13] for further details).

Aiming at minimizing the volume of the confidence regions, in [10], Bishop *et al.* present a summary of the main results for bearing, ranging, TOA and time-difference-of-arrival (TDOA). The sensors are considered to be homogeneous with same variance. The authors provide interesting results presenting the necessary conditions for optimal positions of the sensors in two-dimensional problems.

The optimal positioning of TDOA sensors has also been considered in [12], where the A-optimality criterion was used to optimize the average variance of the estimate for homogeneous sensors in 2D scenarios. A solution considering an extended Kalman filter (EKF) and a nonlinear programming problem to find the optimal trajectories of the sensor illustrate the work with interesting simulation results.

Using a common approach for three different optimization problems, [14] provides an overview on range-only, bearing-only and received signal strength optimal positioning, in two- and three-dimensional spaces, extending the work in [21]. Beyond the unification of the theory for the three methods, the extension of the localization problem to three-dimensional space constitutes the main contributions of the work.

In the absence of theory supporting the optimal placement of TOA sensors in three-dimensional spaces, Ray and Mahajan [16] have proposed a genetic algorithm to solve a geometrically constrained optimization problem, assuming that the position of the sound source is known. The results obtained from simulation are very interesting and are in agreement with the result presented latter on in this paper.

Other works also include the analysis for bearing-only and combined bearing/ranging optimal positioning of sensors in [15] and range-only in [17]. Martinez and Bullo [11] proposed an optimal positioning algorithm for a network of ranging sensors, which uses an EKF and a decentralized control method to drive the sensors towards their optimal positions.

As mentioned in previous works ([14], [16]), the extension of the optimal positioning of sensor to 3D is not trivial. Motivated by marine applications, in this paper we present our results in the context of TOA-only sound source localization in an unconstrained three-dimensional space. The remainder of this paper is organized as follows: The section II presents the main the measurement model and the metric used to infer about the impact of the measurements on the estimate confidence. In section III, we present the main result of this paper and exploit it in a tracking and positioning algorithm presented in section IV. We finally present the simulation results in section V.

II. Background

As seen before, several methods can be used for localization of sound source underwater. We are particularly interested in the TOA problem for which, a minimum of four sensor is required to solve the problem of position estimation. Note that, in addition to the position, the time of emission is unknown. Next, we present the main theoretical background, including the measurement model and the specific FIM used in this paper to find the optimal relative positions of the sensors, with respect to the target, to obtain the best possible estimate, i.e., with the least possible uncertainty.

A. Time-of-arrival measurements

Consider a set of N sensors. We write the vector of observations as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \mathbf{t} + \mathbf{w_s} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix} + \begin{bmatrix} w_{s_1} \\ w_{s_2} \\ \vdots \\ w_{s_N} \end{bmatrix}, \tag{1}$$

where $\mathbf{t} \in \mathbb{R}^N$ is the vector of the times-of-arrival (TOA), which is corrupted by a noise vector $\mathbf{w_s} = [w_{s_1}\ w_{s_2}\ \dots\ w_{s_N}]^T \in \mathbb{R}^N$.

Note that w_{s_i} are variable and depends on several quantities such as the temperature along the water column, which also varies according to the horizontal position, and the positions of both emitter and receivers. Moreover, effects such as multipath or occlusions influence the distribution followed by w_{s_i}. In this paper, we assume that, due to the proximity of the sensors to the target, such effects are negligible and we can model the noise variables to be drawn from a normal distribution $w_{s_i} \sim \mathcal{N}(0, \sigma_i^2)$, where σ_i^2 is the variance. We also assume that the w_{s_i} are uncorrelated.

Let us define the position of the target by the vector $\eta_t = [x_t\ y_t\ z_t]^T$ and the position of the i-th sensor placed on the robots as $\eta_{s_i} = [x_{s_i}\ y_{s_i}\ z_{s_i}]^T$, $i = 1, ..., N$, where $N \geq 4$ in the present case. Therefore, the relative positions are given by

$$\tilde{\eta}_i = \eta_{s_i} - \eta_t,\ i = 1, ..., N. \tag{2}$$

The entries of the vector $\mathbf{t}$ are then given by:

$$t_i = \sqrt{\tilde{\eta}_i^T \tilde{\eta}_i}/c_s + t_t. \tag{3}$$

We can note that $\tilde{\eta}_i$ and t_t are unknown, hence the need for four sensors to determine the three-dimensional position of the target.

B. The Fisher information matrix

We define the measurement likelihood function $f_t\big(\mathbf{z}, \mathbf{t}(\eta_t)\big)$ and determine the entries of the Fisher information matrix $I(\eta_t) \in \mathbb{R}^{4\times4}$, given by (see [10] and [11]) as

$$[I(\eta_t)]_{kl} = E\Big\{ \frac{\partial}{\partial(\eta_t)_k} \ln\Big(f_t\big(\mathbf{z}, \mathbf{t}(\eta_t)\big)\Big) \cdot \frac{\partial}{\partial(\eta_t)_l} \ln\Big(f_t\big(\mathbf{z}, \mathbf{t}(\eta_t)\big)\Big)\Big\}, \tag{4}$$

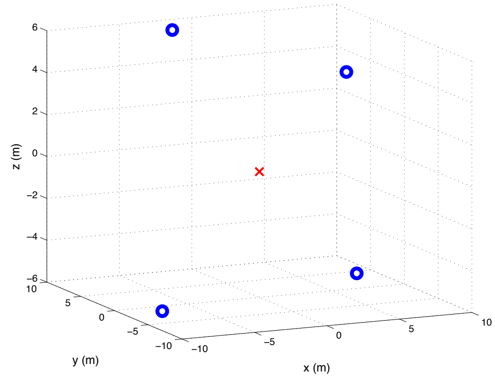

Fig. 1. Optimal positions of receivers (circles). The target (cross) is located at the origin. The parameters considered are $\sigma_i = 10^{-4}$ s $\forall i$.

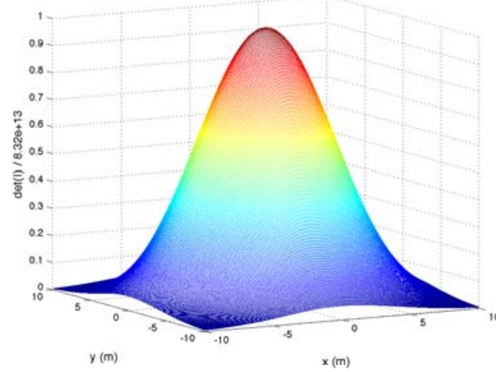

Fig. 2. Metric value as a function of target position as a function of the target position in the plane defined by $z_t = 0$. The parameters considered are $c_s = 1500$ m/s, $\sigma_i = 10^{-4}$ s $\forall i \in \{1, ..., N\}$.

where $E\{\cdot\}$ denotes the expected value and the notation $[\cdot]_{ij}$ is used to identify the matrix entry in the i-th row and j-th column. Similarly, the notation $(\cdot)_i$ is used to denote the i-th element of a vector.

By assuming that the likelihood $f_t\big(\mathbf{z}, \mathbf{t}(\eta_t)\big)$ is a Gaussian function, one can show that the Fisher information matrix results into

$$I(\eta_t) = \big(J_{\mathbf{t}}(\eta_t)\big)^T \Sigma^{-1} J_{\mathbf{t}}(\eta_t), \tag{5}$$

where $J_{\mathbf{t}}(\eta_t)$ is the Jacobian of the $\mathbf{t}(\eta_t)$ vector function and $\Sigma = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2)$ is the covariance matrix.

The Jacobian of $\mathbf{t}(\eta_t)$ can be written as

$$J_{\mathbf{t}}(\eta_t) = -\frac{1}{c_s}\begin{bmatrix} \tilde{\eta}_1^T/||\tilde{\eta}_1|| & c_s \\ \tilde{\eta}_2^T/||\tilde{\eta}_2|| & c_s \\ \vdots & \vdots \\ \tilde{\eta}_N^T/||\tilde{\eta}_N|| & c_s \end{bmatrix}$$

and, after some algebra, we can write the Fisher information matrix as follows:

$$I(\eta_t) = \frac{1}{c_s^2}\begin{bmatrix} \sum_{i=1}^N \frac{\tilde{\eta}_i\tilde{\eta}_i^T}{\sigma_i^2||\tilde{\eta}_i||^2} & -c_s\sum_{i=1}^N \frac{\tilde{\eta}_i}{\sigma_j^2||\tilde{\eta}_i||} \\ -c_s\sum_{i=1}^N \frac{\tilde{\eta}_i^T}{\sigma_i^2||\tilde{\eta}_i||} & \sum_{i=1}^N \frac{c_s^2}{\sigma_i^2} \end{bmatrix}. \tag{6}$$

In the next section, we present the necessary conditions to minimize the volume of the confidence region.

III. Optimal positions of sensors

This section presents the analysis and the conditions to obtain the best achievable performance of an efficient estimator tracking the position of a sound source using TOA measurements. For the sake of illustration, the Fig. (1) and (2) show the optimal positions of the sensors and the corresponding FIM determinant, respectively.

We decompose the Fisher information matrix (6) as

$$I(\eta_t) = \frac{1}{c_s^2}\begin{bmatrix} I_1(\eta_t) & I_2(\eta_t) \\ I_3(\eta_t) & I_4(\eta_t) \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} I_1(\eta_t) &= \sum_{i=1}^N \tilde{\eta}_i\tilde{\eta}_i^T/(\sigma_i^2||\tilde{\eta}_i||^2) \\ I_2(\eta_t) &= I_3(\eta_t)^T = -c_s\sum_{i=1}^N \tilde{\eta}_i/(\sigma_i^2||\tilde{\eta}_i||) \\ I_4(\eta_t) &= c_s^2\sum_{i=1}^N \frac{1}{\sigma_i^2}. \end{aligned}$$

In order to explore the FIM under the relative angular position of the target and of the sensors, we express the matrix in terms of spherical coordinates.

$$\begin{cases} \tilde{x}_i &= ||\tilde{\eta}_i||s\theta_i c\psi_i \\ \tilde{y}_i &= ||\tilde{\eta}_i||s\theta_i s\psi_i \\ \tilde{z}_i &= ||\tilde{\eta}_i||c\theta_i \end{cases}, \tag{8}$$

where $s\,\cdot = \sin(\cdot)$ and $c\,\cdot = \cos(\cdot)$.

The submatrices of the Fisher information matrix thus become

$$\begin{aligned} I_1(\eta_t) &= \sum_{i=1}^N \frac{1}{\sigma_i^2}\begin{bmatrix} s\theta_i^2 c\psi_i^2 & s\theta_i^2 c\psi_i s\psi_i & s\theta_i c\theta_i c\psi_i \\ s\theta_i^2 c\psi_i s\psi_i & s\theta_i^2 s\psi_i^2 & s\theta_i c\theta_i s\psi_i \\ s\theta_i c\theta_i c\psi_i & s\theta_i c\theta_i s\psi_i & c\theta_i^2 \end{bmatrix} \\ I_2(\eta_t) &= I_3(\eta_t)^T = -c_s\sum_{i=1}^N \frac{1}{\sigma_i^2}\begin{bmatrix} s\theta_i c\psi_i \\ s\theta_i s\psi_i \\ c\theta_i \end{bmatrix} \\ I_4(\eta_t) &= c_s^2\sum_{i=1}^N \frac{1}{\sigma_i^2} \end{aligned}$$

It is interesting to note that, under this scenario, the Fisher information matrix does not depend on the range between the sensors and the target but only on the relative angles.

In order to find the optimal position of the sensor to observe the target, we choose the determinant of the Fisher information as the metric M for this scenario:

$$M = det(I(\eta_t)). \tag{9}$$

This metric quantifies the amount of information given by a set of measurements collected by the sensors. Our objective now is to maximize the amount of information. Therefore our problem is reduced to finding the set of angles θ_i and ψ_i that maximizes M. We use the D-optimality criterion to derive the optimal positions of the sensors, that is, we minimize the volume of the uncertainty ellipsoid [13]. Mathematically, we want to find the pairs

$$\{\theta_i^*,\ \psi_i^*\} = \underset{\substack{\{\theta_i,\ \psi_i\}\\ i=1,\ldots,N}}{\operatorname{argmax}}\ M(\{\theta_i,\ \psi_i\}). \quad (10)$$

The main result of this paper is summarized next.

Theorem 1: Consider a set of N TOA sensors with variances $\{\sigma_i^2\}_{i=1}^N$ and relative orientations $\{\theta_i, \psi_i\}_{i=1}^N$, with respect to the sound source position. The determinant of the Fisher information matrix, $M(\cdot)$, is maximal and equal to $M^* = \frac{1}{27c_s^6}\Big(\sum_{i=1}^N \frac{1}{\sigma_i^2}\Big)^4$ when the following conditions are simultaneously satisfied:

$$\sum_{i=1}^N \frac{s\theta_i c\psi_i}{\sigma_i^2} = 0, \quad \sum_{i=1}^N \frac{s\theta_i s\psi_i}{\sigma_i^2} = 0, \quad \sum_{i=1}^N \frac{c\theta_i}{\sigma_i^2} = 0, \quad (11)$$

$$\sum_{i=1}^N \frac{s2\theta_i c\psi_i}{\sigma_i^2} = 0, \quad \sum_{i=1}^N \frac{s2\theta_i s\psi_i}{\sigma_i^2} = 0, \quad (12)$$

$$\sum_{i=1}^N \frac{s\theta_i^2 c2\psi_i}{\sigma_i^2} = 0, \quad \sum_{i=1}^N \frac{s\theta_i^2 s2\psi_i}{\sigma_i^2} = 0, \quad (13)$$

$$\sum_{i=1}^N \frac{c\theta_i^2}{\sigma_i^2} = \frac{1}{3}\sum_{i=1}^N \frac{1}{\sigma_i^2}. \quad (14)$$

Proof: We first note that the Fisher information matrix is symmetric. The metric, M, i.e., the determinant of the matrix can be written in an alternative form as

$$det\big(I(\{\theta_i, \psi_i\})\big) = \frac{1}{c_s^8} det(I_4) det(I_1 - I_2 I_4^{-1} I_3) \quad (15)$$

where the arguments of the matrix I_k, $k = 1, ..., 4$, were dropped for clarity.

Since I_4 is scalar and positive definite, we note that $I_2 I_4^{-1} I_3 \geq 0$ (the proof is simply based on the fact that, for any $x \in \mathbb{R}^3$, $x^T I_2 I_4^{-1} I_3 x = I_4^{-1} x^T I_2 I_2^T x = I_4^{-1}(x^T I_2)^2 \geq 0$ which, in turn, implies that $I_2 I_4^{-1} I_3$ is positive semi-definite) and therefore

$$\begin{aligned} M(\{\theta_i, \psi_i\}) &= \frac{1}{c_s^8} det(I_4) det(I_1 - I_2 I_4^{-1} I_3) \\ &\leq \frac{1}{c_s^8} det(I_4) det(I_1). \end{aligned} \quad (16)$$

Hence, we can conclude that the equality is achieved when $I_2 = I_3^T = \mathbf{0}$ and prove (11).

We focus now our attention on the determinant of I_1. Similarly to the previous step, we decompose the matrix in blocks as follows:

$$I_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (17)$$

with

$$A = \sum_{i=1}^N \frac{1}{\sigma_i^2} \begin{bmatrix} s\theta_i^2 c\psi_i^2 & s\theta_i^2 c\psi_i s\psi_i \\ s\theta_i^2 c\psi_i s\psi_i & s\theta_i^2 s\psi_i^2 \end{bmatrix},$$

$$B = C^T = \sum_{i=1}^N \frac{1}{\sigma_i^2} \begin{bmatrix} s\theta_i c\theta_i c\psi_i \\ s\theta_i c\theta_i s\psi_i \end{bmatrix},$$

$$D = \sum_{i=1}^N \frac{1}{\sigma_i^2} c\theta_i^2.$$

Under the assumption that $I_2 = I_3^T = \mathbf{0}$, we can write

$$M(\{\theta_i, \psi_i\}) = \frac{1}{c_s^8} det(I_4) det(D) det(A - BD^{-1}C) \quad (18)$$

Using the same reasoning as above and noting that D is positive semi-definite and invertible, we have $BD^{-1}C \geq 0$. Therefore, the choice of

$$B = C^T = \sum_{i=1}^N \frac{1}{\sigma_i^2} \begin{bmatrix} s2\theta_i c\psi_i \\ s2\theta_i s\psi_i \end{bmatrix} = \mathbf{0}$$

maximizes the metric M over $B = C^T$. Note that we used the fact that $s\theta_i c\theta_i = s2\theta_i$. This proves the necessity of (12). Under this condition, the metric M becomes

$$\begin{aligned} M(\{\theta_i, \psi_i\}) &= \frac{1}{c_s^8} det(I_4) det(D) det(A) \\ &= \frac{1}{c_s^8} I_4 D det(A) \end{aligned} \quad (19)$$

We concentrate now on maximizing $det(A)$. Using the equalities $c\psi_i^2 = (1 + c2\psi_i)/2$ and $s\psi_i^2 = (1 - c2\psi_i)/2$, we rewrite the matrix A as

$$A = \sum_{i=1}^N \frac{1}{\sigma_i^2} \begin{bmatrix} s\theta_i^2(\frac{1}{2} + \frac{c2\psi_i}{2}) & s\theta_i^2 c\psi_i s\psi_i \\ s\theta_i^2 c\psi_i s\psi_i & s\theta_i^2(\frac{1}{2} - \frac{c2\psi_i}{2}) \end{bmatrix}$$

whose determinant becomes

$$\begin{aligned} det(A) = &\Big(\sum_{i=1}^N \frac{1}{2\sigma_i^2} s\theta_i^2\Big)^2 - \Big(\sum_{i=1}^N \frac{1}{2\sigma_i^2} \frac{s\theta_i^2 c2\psi_i}{2}\Big)^2 \\ &- \Big(\sum_{i=1}^N \frac{1}{2\sigma_i^2} s\theta_i^2 s2\psi_i\Big)^2 \end{aligned} \quad (20)$$

Hence, setting $\sum_{i=1}^N \frac{1}{2\sigma_i^2} \frac{s\theta_i^2 c2\psi_i}{2} = \sum_{i=1}^N \frac{1}{2\sigma_i^2} s\theta_i^2 s2\psi_i = 0$ maximizes the determinant of A over these terms and verifies the necessity of (13).

Finally, if the conditions (11)-(13) are verified, the metric is given by

$$M = \frac{1}{c_s^8} c_s^2 \sum_{i=1}^N \frac{1}{\sigma_i^2} \sum_{i=1}^N \frac{1}{\sigma_i^2} c\theta_i^2 \Big(\sum_{i=1}^N \frac{1}{2\sigma_i^2} s\theta_i^2\Big)^2,$$

which can be rewritten as

$$M = \frac{1}{c_s^8} c_s^2 \sum_{i=1}^N \frac{1}{\sigma_i^2} \sum_{i=1}^N \frac{1}{\sigma_i^2} c\theta_i^2 \Big(\sum_{i=1}^N \frac{1}{2\sigma_i^2} - \sum_{i=1}^N \frac{1}{2\sigma_i^2} c\theta_i^2\Big)^2. \quad (21)$$

We define $\xi = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} c\theta_i^2$ and the constant $\alpha = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}$ and write, by substitution,

$$M = \frac{1}{c_s^6} \alpha\xi\Big(\frac{1}{2}\alpha - \frac{1}{2}\xi\Big)^2.$$

Note that $\xi \in [0, \alpha]$. It can be shown (by finding the roots of the derivative with respect to ξ) that the maximal value of M is reached when $\xi = \frac{\alpha}{3}$, or equivalently

$$\sum_{i=1}^{N} \frac{1}{\sigma_i^2} c\theta_i^2 = \frac{1}{3} \sum_{i=1}^{N} \frac{1}{\sigma_i^2},$$

which is the same condition as in (14). This ends the proof. ■

IV. Positioning and estimation

In the previous section, we have considered that the target position is perfectly known, which is very unlikely in most of the real estimation problems. In this section, we propose an algorithm for estimation and positioning. The estimation algorithm is based on a well-known nonlinear Newton's method to solve a nonlinear least-square formulated problem.

The algorithm implemented is composed of two sequential steps that are iteratively ran whenever a new measurement set is available:

1) Estimation: Given a set of TOA measurements, the position of the target is estimated using a least-squares method. Since the problem is nonlinear, we implement a Newton's method which, for each new measurement, iterates until the difference between the new estimate and the previous is less than a threshold value or exceeds a maximum number of iterations;
2) Positioning: Given the estimate, the optimal positions of the receivers are moved towards their (estimated) optimal relative positions. We use a gradient descent algorithm that iteratively computes the directions to follow and stops after a predefined number of iterations.

We do not address the convergence issues in this paper, which have a large coverage in specialized literature. Nevertheless, we anticipate that the overall speed performances and number of iterations that each algorithm requires to obtain the same estimate and positioning depend on the gains used in the Newton's and gradient descent methods.

A. Estimation

In order to estimate the target position, we have formulated the estimation problem as a nonlinear least-square problem. The Newton's method is then applied to find the optimal point that minimizes a given error function.

We define the state estimate

$$\hat{X}_k = \left[\hat{\eta}_t(k)^T \ \hat{t}_t(k)\right]^T \in \mathbb{R}^4 \tag{22}$$

as the concatenation of the estimated position vector and the time of emission, where $k \in \mathbb{N}$.

Recall that the relative positions of the receivers are given by $\tilde{\eta}_i = \eta_{s_i} - \eta_t$ and that the times of arrival are given by $\mathbf{t} = [t_1, ..., t_N]$ and define the *error* vector as

$$g(\eta_t, t_t) = \begin{bmatrix} ||\tilde{\eta}_1||^2 - \nu_s^2(t_1 - t_t)^2 \\ \vdots \\ ||\tilde{\eta}_N||^2 - \nu_s^2(t_N - t_t)^2 \end{bmatrix}. \tag{23}$$

Given an initial guess estimate of the state estimate $\hat{X}_0$ and a set of measurements composing the entries of $\mathbf{z} = [z_1, ..., z_N]$, the state estimate is recursively estimated, until a given criterion is met, by the Newton's method applied to multidimensional nonlinear equations:

$$\hat{X}_{k+1} = \hat{X}_k - K(J_g(\hat{X}_k))^{\dagger} g\left(\hat{\eta}_t(k), \hat{t}_t(k)\right), \tag{24}$$

where $K > 0$ is a scalar gain, $J_g(\hat{X}_k)$ stands for the Jacobian of g evaluated at $\hat{X}_k$ for the set of measured times of arrival, which is given by

$$J_g(\hat{X}_k) = 2 \cdot \begin{bmatrix} -\tilde{\eta}_1^T & c_s^2(t_1 - t_t) \\ \vdots & \vdots \\ -\tilde{\eta}_N^T & c_s^2(t_N - t_t) \end{bmatrix}_{|\tilde{\eta}_i = \eta_{s_i} - \hat{\eta}_t(k), t_t = \hat{t}_t(k), t_i = z_i},$$

and $(\cdot)^{\dagger}$ denotes the pseudo-inverse.

B. Positioning algorithm

A very simple positioning algorithm can be obtained by applying a gradient descent algorithm to a function that takes into account the individual errors of the constraints in (11)-(14). For this purpose, we define the error functions e_i as the square of the conditions above, i.e. $e_1 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s\theta_i c\psi_i\right)^2$, $e_2 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s\theta_i s\psi_i\right)^2$, $e_3 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} c\theta_i\right)^2$, $e_4 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s2\theta_i c\psi_i\right)^2$, $e_5 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s2\theta_i s\psi_i\right)^2$, $e_6 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s\theta_i^2 c2\psi_i\right)^2$, $e_7 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} s\theta_i^2 s2\psi_i\right)^2$, $e_8 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2} c\theta_i^2 - \frac{1}{3}\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right)^2$.

We formulate the problem using a potential function

$$V(\eta_{s_1}, ..., \eta_{s_N}) = \sum_{i=1}^{8} e_i(\eta_{s_1}, ..., \eta_{s_N}), \tag{25}$$

which is used to iteratively update the receiver positions using the straightforward gradient descent algorithm:

$$\begin{bmatrix} \eta_{s_1}(l+1) \\ \vdots \\ \eta_{s_N}(l+1) \end{bmatrix} = \begin{bmatrix} \eta_{s_1}(l) \\ \vdots \\ \eta_{s_N}(l) \end{bmatrix} + K_V \begin{bmatrix} \nabla_{\eta_{s_1}} V \\ \vdots \\ \nabla_{\eta_{s_N}} V \end{bmatrix}_{\big|\eta_{s_i} = \eta_{s_i}(l)}, \tag{26}$$

where $K_V > 0$ is a scalar gain and $\nabla_v(\cdot)$ denotes the gradient of a function with respect to a vector v.

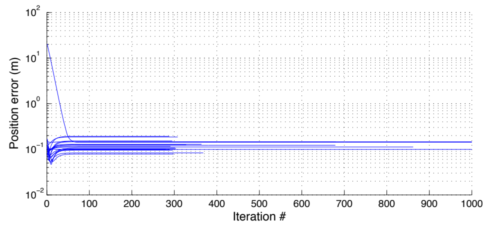

Fig. 3. Error evolution

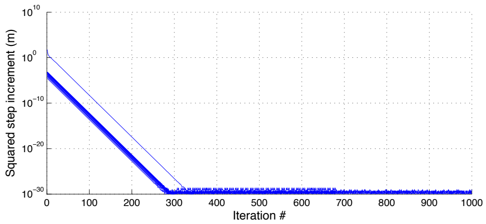

Fig. 4. Step increment

V. Results

We have conducted simulations to validate our approach and assess its robustness. For the results presented next, we assumed that the variances are equal for all the sensors, i.e. $\sigma_i = \sigma$ for all i, and we have considered the gains $K = 10^{-1}$ and $K_V = 10^{-1} \cdot \sigma^4$.

The initial positions of the receivers were generated randomly within a cube of side equal to 100 meters. To ensure that the position is observable, we have considered the initial configurations that verify $M \geq \underline{M}$ only. Otherwise, the estimation algorithm can originate poor estimates, which could lead to divergence. For the simulations, we have set the noise distribution to be uniform in the interval $[-\sigma, \sigma]$. Two cases were considered: $\sigma = 10^{-4}$ s and $\sigma = 10^{-2}$ s. In underwater environments, the former value is realistic when using digital signal processing. The latter was exaggerated in order to assess the robustness of the method.

For each *epoch*, a new set of times of arrival is drawn, which is subsequently used to estimate the position of the target. For what concerns optimal placement of sensors, the positioning algorithm is iterated five times in each epoch, after the estimate has been computed. The recursive estimation algorithm is stopped after the sum of the increment squares verifies $(K(J_g(\hat{X}_k))^\dagger g)^T(K(J_g(\hat{X}_k))^\dagger g) < 10^{-30}$ (see (24)) or has been iterated more than one thousand times. These values have been selected according to practice and are justified in Fig. (3) and Fig. (4), where one can see the evolution of the estimate error and the respective estimate increment for a simulation considering $\sigma = 10^{-4}$ s. In Fig. (3), the plot shows that most of the estimation steps do not use more than five hundred iterations because the increment becomes small and satisfies the first condition.

The Fig. 5 depicts the trajectories (lines) and the final positions (circles) of the mobile receivers for $\sigma = 10^{-4}$ s. The cross (at the origin) indicates the position of the sound source. The trajectories are smooth and remain very stable after tracking the sound source. In Fig. 8, we show the position error resulting from the estimation algorithm. The estimate error rapidly decreases from the first to the second epoch and remains around 0.1 meters.

The Fig. 7 shows the evolution of the trajectories of the receivers. The lines display coarser trajectories than in the previous case. This is the result of a larger noise that originates coarser estimates that vary more significantly. The optimal positions of the sensor are adjusted after each new estimate and consequently become more noisy. Larger relative distances would make the trajectory smoother as the sensitivity of a change on the estimate would become smaller, i.e., the corresponding angles $\{\theta_i, \psi_i\}$ would suffer less from variations on estimate.

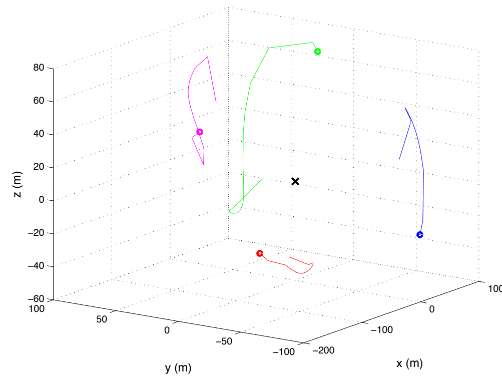

Fig. 5. Optimal positioning

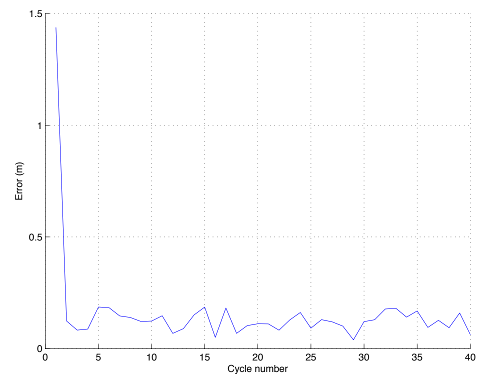

Fig. 6. Estimate error

VI. Conclusions

In this paper, we have presented the solution for optimal placement of TOA sensors in three-dimensional spaces. Based on the FIM, we have derived the necessary conditions to minimize the volume of the confidence region and consequently improve the estimate. Using the geometric properties of the optimal sensor positions, we illustrated the results obtained in this paper using a two-step algorithm that sequentially considers estimation and placement. The simulation demonstrated the robustness of the approach and leaves space for

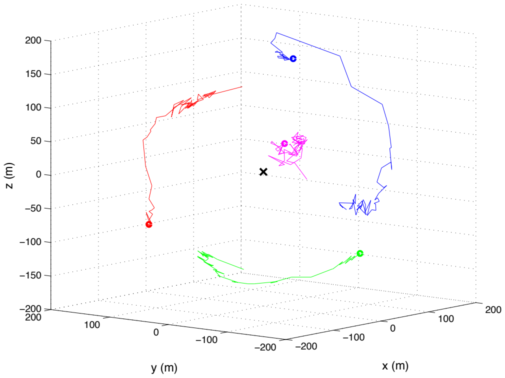

Fig. 7. Optimal positioning

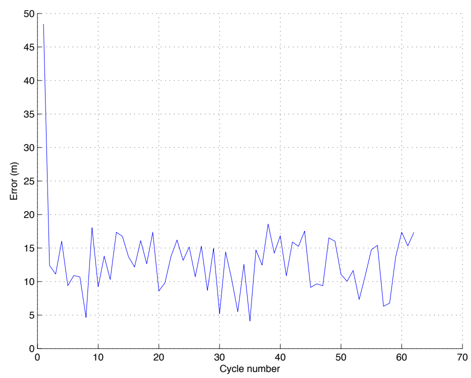

Fig. 8. Estimate error

further improvements and analysis. The implementation of this approach into real robots would require special attention on sensor motions. In particular, the limited speed of the vehicles and the precision on positioning would constitute the main constraints. A proper tuning of gain and/or saturating the position increments would surely solve the former. The second constraint implies that robotic platform equipped with precise navigation sensors and control laws have to be employed to improve the tracking process.

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