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ANOSOV DIFFEOMORPHISMS

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Abstract. We use Adler, Tresser and Worfolk decomposition of Anosov automorphisms to give an explicit construction of the stable and unstable C^{1+} self-renormalizable sequences.

1. **Introduction.** Several authors have studied the existence of correspondences between smooth conjugacy classes of Anosov diffeomorphisms and smooth self-renormalizable structures, cohomology classes of Hölder cocycles, scaling functions, ratio functions and eigenvalues (see [2, 5, 10, 11, 13, 16, 17, 21–33]).

In [2], it is presented an explicit construction of the self-renormalizable structures for Anosov diffeomorphisms that are topologically conjugate to the golden Anosov automorphism. Here, using the Adler, Tresser and Worfolk [1] decomposition of Anosov automorphisms, we extend the explicit construction of self-renormalizable structures to self-renormalizable sequences that apply to Anosov diffeomorphisms of any given topological class. The one-dimensional smooth self-renormalizable sequences encode all the smooth information of the foliations of Anosov diffeomorphisms. This work extends the results of Pinto and Rand for hyperbolic diffeomorphisms on surfaces, presented in Chapter 4 of [32], to self-renormalizable sequences.

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2. **Anosov diffeomorphisms.** Let

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

be a hyperbolic matrix, i.e. the eigenvalues λ and μ of $\tilde{A}$ are such that $|\lambda| > 1$ and $|\lambda\mu| = 1$. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and let $\pi_{\mathbb{T}} : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the natural projection. The matrix $\tilde{A}$ determines a unique *Anosov automorphism* $A : \mathbb{T} \to \mathbb{T}$ such that

$$A \circ \pi_{\mathbb{T}} = \pi_{\mathbb{T}} \circ \tilde{A}.$$

By Williams [39] and Adler, Tresser and Worfolk [1], there exist $N \in \mathbb{N}$, an invertible matrix C, and

$$\tilde{G} = \left[\begin{pmatrix} 0 & 1 \\ 1 & a_{N-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \right],$$

such that $\tilde{G} = C\tilde{A}C^{-1}$, if $\lambda > 1$, and $-\tilde{G} = C\tilde{A}C^{-1}$, if $-\lambda > 1$. For simplicity of notation, from now on, we will consider the indices mod N. We note that the modulus of the contracting eigenvalue μ is equal to $|\mu| = \prod_{i=0}^{N-1} \gamma_i$, where $\gamma_i = 1/(a_i + 1/(a_{i-1} + 1/\ldots))$. Let $\vec{v}_i = (-1, \gamma_{i-2})$, $\vec{w}_i = (\gamma_{i-1}, 1)$ and

$$\mathbb{T}_i = \mathbb{R}^2/(\vec{v}_i\mathbb{Z} \times \vec{w}_i\mathbb{Z}).$$

Let $\pi_i : \mathbb{R}^2 \to \mathbb{T}_i$ be the natural projection and let us define $F_i : \mathbb{T}_i \to \mathbb{T}_{i+1}$ by

$$F_i(\pi_i(x\vec{v}_i + y\vec{w}_i)) = \pi_{i+1}(y\vec{v}_{i+1} + (x + a_i y)\vec{w}_{i+1}). \tag{1}$$

Hence, the map $F_i : \mathbb{T}_i \to \mathbb{T}_{i+1}$ has the following diagonal form

$$F_i(\pi_i(x\vec{i} + y\vec{j})) = \pi_{i+1}(-\gamma_i x\vec{i} + \gamma_i^{-1} y\vec{j}),$$

with respect to the canonical basis $< \vec{i}, \vec{j} >$ of $\mathbb{R}^2$. Let

$$G_i = G_{(1,\gamma,N,i)} : \mathbb{T}_i \to \mathbb{T}_i$$

be the Anosov automorphism given by

$$G_i = F_{i-1} \circ \ldots \circ F_0 \circ F_N \circ \ldots \circ F_{i+1} \circ F_i.$$

Let

$$G_{(-1,\gamma,N,i)} : \mathbb{T}_i \to \mathbb{T}_i$$

be the Anosov automorphism given by

$$G_{(-1,\gamma,N,i)} = -G_{(1,\gamma,N,i)}.$$

Therefore, by Adler, Tresser and Worfolk [1], the hyperbolic automorphism A is topologically conjugate to the Anosov automorphim $G_{(1,\gamma,N,0)}$, if $\lambda > 1$, and to the Anosov automorphim $G_{(-1,\gamma,N,0)}$, if $-\lambda > 1$.

Definition 2.1. Let the *ATW-triple* (θ, γ, N) be formed by (i) $\theta \in \{-1, 1\}$, (ii) $N \in \mathbb{N}$ and (iii) $\gamma < 1$ with a m-periodic continuous fraction expansion such that $mk = N$ for some $k \in \mathbb{N}$. Let $\mathcal{ATW}$ be the set of all ATW-triples.

We say that $\mathcal{S}$ is a C^{1+} *structure on* $\mathbb{T}$ if, for every pair of charts contained in $\mathcal{S}$, there is $\alpha > 0$ such that the overlap map is a $C^{1+\alpha}$ diffeomorphism. A triple $(G, \mathbb{T}, \mathcal{S})$ is a C^{1+} *Anosov diffeomorphism* if there is $\alpha > 0$ and a finite $C^{1+\alpha}$ atlas $\mathcal{S}' \subset \mathcal{S}$ of the C^{1+} structure $\mathcal{S}$ on $\mathbb{T}$ such that (i) G is a $C^{1+\alpha}$ diffeomorphism with respect to the atlas $\mathcal{S}'$; (ii) the tangent bundle has a $C^{1+\alpha}$ uniformly hyperbolic splitting into a stable direction and an unstable direction with respect to the atlas $\mathcal{S}'$ (see [34]). Let $\mathcal{G}$ be the set of all C^{1+} Anosov diffeomorphisms.

By Franks [7] and Manning [14], every C^{1+} Anosov diffeomorphism $(G, \mathbb{T}, \mathcal{S})$ is topologically conjugate to a unique Anosov automorphism $G_{(\theta,\gamma,N,0)}$. Let the map $T : \mathcal{G} \to \mathcal{ATW}$ be defined by $T(G) = (\theta, \gamma, N)$, where (θ, γ, N) is such that G is topologically conjugate to $G_{(\theta,\gamma,N,0)}$. Hence, the map T determines a one-to-one correspondence between ATW-triples and topological conjugacy classes of Anosov automorphisms. Therefore, the topological classification of Anosov automorphisms is completely determined by the ATW-triples.

We denote by $\mathcal{G}(\theta, \gamma, N)$ the set of all C^{1+} Anosov diffeomorphisms $(G, \mathbb{T}, \mathcal{S})$ that are topologically conjugate to the Anosov automorphism $G_{(\theta,\gamma,N,0)}$.

Let $\mathcal{S}(\theta, \gamma, N)$ be the set of all C^{1+} structures $\mathcal{S}$ for which the triples $(G_{(\theta,\gamma,N,0)}, \mathbb{T}_0, \mathcal{S})$ are C^{1+} Anosov diffeomorphisms. We observe that the natural projection $\pi_0 : \mathbb{R}^2 \to \mathbb{T}_0$ determines a unique *affine canonical structure* $\underline{\mathcal{S}} \in \mathcal{S}(\theta, \gamma, N)$ with the property that the map $G_{(\theta,\gamma,N,0)}$ is an Anosov automorphism with respect to the canonical structure $\underline{\mathcal{S}}$.

Lemma 2.2. *The map* $\mathbf{S} : \mathcal{S}(\theta, \gamma, N) \to \mathcal{G}(\theta, \gamma, N)$ *that associates to each* C^{1+} *structure* $\mathcal{S}$ *the* C^{1+} *conjugacy class of* $(G(\theta, \gamma, N, 0), \mathbb{T}_0, \mathcal{S})$ *induces a one-to-one correspondence between* C^{1+} *structures and* C^{1+} *conjugacy classes.*

See the proof of Lemma 2.2 in [22].

Hence, given a topological conjugacy class $\mathcal{G}(\theta, \gamma, N)$, to characterize all the C^{1+} conjugacy classes in $\mathcal{G}_{(\theta,\gamma,N,0)}$ it is enough to characterize all the C^{1+} structures $\mathcal{S}$ with the property that the triple $(G_{(\theta,\gamma,N,0)}, \mathbb{T}_0, \mathcal{S})$ is a C^{1+} Anosov diffeomorphism.

Remark 1. Hence, from now on, we fix the map $G = G_{(\theta,\gamma,N,0)} : \mathbb{T}_0 \to \mathbb{T}_0$ and we consider all the structures $\mathcal{S}$ on $\mathbb{T}_0$ such that $(G, \mathbb{T}_0, \mathcal{S})$ is a C^{1+} Anosov diffeomorphism.

Given a C^{1+} Anosov diffeomorphism $(G_{(\theta,\gamma,N,0)}, \mathbb{T}_0, \mathcal{S})$, let $\mathcal{S}_i$ be the pushforward of the C^{1+} structure $\mathcal{S}$ by $F_{i-1} \circ \cdots \circ F_1 \circ F_0$, i.e. $\mathcal{S}_i = (F_{i-1} \circ \cdots \circ F_1 \circ F_0)_* \mathcal{S}$ where $F_0, F_1, \cdots, F_{i-1}$ are defined by (1). We note that $\mathcal{S}_0 = \mathcal{S}$. By construction, $F_i : \mathbb{T}_i \to \mathbb{T}_{i+1}$ is a C^{1+} diffeomorphism with respect to the C^{1+} structures $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ and so $G_i = G_{(\theta,\gamma,N,i)}$ is a C^{1+} Anosov diffeomorphism with respect to the C^{1+} structure $\mathcal{S}_i$.

2.1. **Smooth foliations.** Let $d_{i,\mathcal{S}_i}$ be the distance on the torus $\mathbb{T}_i$ determined by a Riemannian metric $\rho(\mathcal{S}_i)$ compatible with the structure $\mathcal{S}_i$. Given $z \in \mathbb{T}_i$, we denote the *local stable manifold* $W_i^s(z, \varepsilon_0)$ *through* z by

$$W_i^s(z, \varepsilon_0) = \{w \in \mathbb{T}_i : d_{i,\mathcal{S}_i}(G_i^n(w), G_i^n(z)) \le \varepsilon_0, \text{ for all } n \ge 0\}.$$

By the Stable Manifold Theorem (see [34]), the images of the local stable manifolds by the charts of the structure $\mathcal{S}_i$ are smooth curves.

The *stable manifold* $W_i^s(\pi_i(x, y))$ passing through $\pi_i(x, y)$ is the projection $W_i^s(z) = \pi_i(H)$ by π_i of the horizontal line H in the plane passing through (x, y). An *open* (resp. *closed*) *stable leaf segment* I passing through a point $\pi_i(x, y)$ is of the form $I = \pi_i(K, y)$, where K is an open (resp. closed) interval in $\mathbb{R}$ with $x \in \mathrm{int}(K)$. The *endpoints* of the stable leaf segment I are the points $\pi_i(\partial K, y)$. A *stable leaf segment* is either an open or closed stable leaf segment. The *interior* of a stable leaf segment is the complement of its boundary. A map $c : I \to \mathbb{R}$ is a *stable leaf chart* of a stable leaf segment I if c is a homeomorphism onto its image. If I is a stable leaf segment of $\mathbb{T}_i$ then $F_i(I)$ is a stable leaf segment of $\mathbb{T}_{i+1}$ and $F_{i-1}^{-1}(I)$ is a stable leaf segment of $\mathbb{T}_{i-1}$. Similarly, the Stable Manifold Theorem implies that the images of the stable manifolds by the charts of the structure $\mathcal{S}_i$ are smooth curves. We note that the unstable manifolds of G_i are the projection by π_i of the vertical lines in the plane. The unstable leaf segments and unstable leaf charts are defined similarly.

Let I and J be open intervals in $\mathbb{R}$. The closure of the set $\pi_i(I \times J)$ is a *rectangle* if $\pi_i|I \times J$ is a homeomorphism onto its image. If $W_i^u(w,\varepsilon)$ and $W_i^s(z,\varepsilon)$ intersect in a unique point $v = W_i^u(w,\varepsilon) \cap W_i^s(z,\varepsilon)$, we denote the intersection point v by $[w,z]$. Given a rectangle R_i in $\mathbb{T}_i$ then, for every $x \in R_i$, there are closed unstable and stable leaf segments $\ell_i^u(x,R_i)$ and $\ell_i^s(x,R_i)$, passing through x, such that $R_i = [\ell_i^u(x,R_i), \ell_i^s(x,R_i)]$, i.e. R_i is the set of all points of the form $[w,z]$ with $w \in \ell_i^u(x,R_i)$ and $z \in \ell_i^s(x,R_i)$. The leaf segments $\ell_i^s(x,R_i)$ and $\ell_i^u(x,R_i)$ are called *spanning stable* and *spanning unstable leaf segments*, respectively.

Definition 2.3. A *basic stable holonomy* is the homeomorphism $\theta_{i,s} : \ell_i^s(x,R_i) \to \ell_i^s(z,R_i)$ defined by $\theta_{i,s}(w) = [w,z]$.

The *basic unstable holonomies* are defined similarly. The *stable lamination atlas $\mathcal{L}_s(\mathcal{S}_i)$, determined by a Riemannian metric* $\rho(\mathcal{S}_i)$ compatible with the C^{1+} structure $\mathcal{S}_i$, is the set of all maps $e_i : I_i \to \mathbb{R}$, where e_i is an isometry between the induced Riemannian metric on the stable leaf segment I_i and the Euclidean metric on the reals. We call the maps $e_i \in \mathcal{L}_s(\mathcal{S}_i)$ the *stable lamination charts*. The *unstable lamination atlas* $\mathcal{L}_u(\mathcal{S}_i)$ is defined similarly. By Theorem 2.1 in [26], the basic stable and unstable holonomies are C^{1+} with respect to the lamination atlases $\mathcal{L}_s(\mathcal{S}_i)$ and $\mathcal{L}_u(\mathcal{S}_i)$, respectively. The affine canonical structures $\underline{\mathcal{S}}_i$ determine canonical affine stable and unstable lamination atlases $\mathcal{L}_s(\underline{\mathcal{S}}_i)$ and $\mathcal{L}_u(\underline{\mathcal{S}}_i)$, respectively.

3. **Self-renormalizable sequences.** Train-tracks are optimal leaf-quotient spaces on which the stable and unstable Markov maps induced by the action of C^{1+} Anosov diffeomorphisms on leaf segments are local homeomorphisms.

Let the *stable leaf box* $t^s_{R_i}$ of a rectangle R_i be the set of all spanning unstable leaf segments of R_i. By the local product structure, one can identify a stable leaf box $t^s_{R_i}$ with any spanning stable leaf segment $\ell_i^s(x,R_i)$ of R_i, i.e. there is a one-to-one correspondence between points y in $\ell_i^s(x,R_i)$ and spanning unstable leaf segments $\ell_i^u(y,R_i)$ in $t^s_{R_i}$. Given a rectangle partition $\{R_1,\ldots,R_n\}$ of the torus, we say that two points $I \in t^s_{R_i}$ and $J \in t^s_{R_j}$ are *equivalent* $I \sim J$, if (i) the unstable leaf segments I and J are unstable boundaries of the rectangles R_i and R_j, (ii) $\mathrm{int}(I \cap J) \neq \emptyset$ and (iii) $R_i \neq R_j$.

The *pre-Markov partition* of G_i is given by $\mathcal{M}_i = \{A_{i,1}, A_{i,2}, B_i\}$, where $A_{i,1} = \pi_i([0,\gamma_{i-1}] \times [0,1])$, $A_{i,2} = \pi_i([\gamma_{i-1},1] \times [0,1])$ and $B_i = \pi_i([-\gamma_{i-1},0] \times [0,\gamma_{i-2}])$. The *stable train-track* T_i^s is the quotient space given by

$$\mathrm{T}_i^s = t^s_{A_{i,1}} \bigsqcup t^s_{A_{i,2}} \bigsqcup t^s_{B_i} / \sim .$$

The *unstable train-track* T_i^u is defined similarly. There is a canonical projection map $\pi_{\mathrm{T}_i^s} : \bigsqcup_{R_i \in \mathcal{M}_i} R_i \to \mathrm{T}_i^s$ that sends the point $x \in R_i$ to the point $\ell^u(x,R_i)$ in T_i^s. A *topologically regular point* I in T_i^s is a point with a unique preimage under $\pi_{\mathrm{T}_i^s}$ (i.e. the preimage of I is not a union of distinct unstable boundaries of pre-Markov rectangles). If a point has more than one preimage by $\pi_{\mathrm{T}_i^s}$, then we call it a *junction.* We note that there is only one junction j_i^s consisting of the union of all unstable boundaries of the *ATW* rectangles A_i and B_i.

A chart $i : I \to \mathbb{R}$ in $\mathcal{L}_s(\mathcal{S}_i)$ determines a *train-track chart* $i_{\mathrm{T}} : I_T \to \mathbb{R}$ for I_T given by $i_T \circ \pi_{\mathrm{T}_i^s} = i$. We denote by $\mathcal{B}'_s(\mathcal{S}_i)$ the set of all train-track charts i_T determined by charts i in $\mathcal{L}_s(\mathcal{S}_i)$. Given any train-track charts $i_T : I_T \to \mathbb{R}$ and $j_T : J_T \to \mathbb{R}$ in $\mathcal{B}'_s(\mathcal{S}_i)$, the overlap map $j_T \circ i_T^{-1}$ is equal to $j \circ \theta_i \circ i^{-1}$, where $i = i_T \circ \pi_{\mathrm{T}_i^s} : I \to \mathbb{R}$ and $j = j_T \circ \pi_{\mathrm{T}_i^s} : J \to \mathbb{R}$ are charts in $\mathcal{L}_s(\mathcal{S}_i)$, and

$$\theta_i : i^{-1}(i_T(I_T \cap J_T)) \to j^{-1}(j_T(I_T \cap J_T))$$

is a basic stable holonomy. By Theorem 2.1 in Pinto and Rand [26], there exists $\alpha > 0$ such that, for all train-track charts i_T and j_T in $\mathcal{B}'_s(\mathcal{S}_i)$, the overlap maps $j_T \circ i_T^{-1} = j \circ \theta_i \circ i^{-1}$ have $C^{1+\alpha}$ diffeomorphic extensions with a uniform bound in the $C^{1+\alpha}$ norm. Hence, $\mathcal{B}'_s(\mathcal{S}_i)$ is a $C^{1+\alpha}$ atlas on T_i^s. Let $\mathcal{B}_s(\mathcal{S}_i)$ be the C^{1+} structure that is C^{1+} compatible with the $C^{1+\alpha}$ atlas $\mathcal{B}'_s(\mathcal{S}_i)$.

The *(stable) Markov map* $m_{i,s} : \mathrm{T}_i^s \to \mathrm{T}_{i-1}^s$ is the mapping induced by the action of F_i on spanning unstable leaf segments and it is defined as follows: if $I \in \mathrm{T}_i^s$, then $m_{i,s}(I) = \pi_{\mathrm{T}_i^s}(F_i(I))$ is the spanning unstable leaf segment containing $F_i(I)$. This map $m_{i,s}$ is a local homeomorphism because F_i sends short stable leaf segments homeomorphically onto short stable leaf segments. The *(unstable) Markov map* $m_{i,u} : \mathrm{T}_i^u \to \mathrm{T}_{i+1}^u$ is defined similarly.

Let us define $m_{1,i,s} = m_{i,s} : \mathrm{T}_i^s \to \mathrm{T}_{i-1}^s$ and, recursively, let $m_{n+1,i,s} = m_{i-n,s} \circ m_{n,i,s} : \mathrm{T}_i^s \to \mathrm{T}_{i-1}^s$. A *$n$-cylinder of* T_i^s is a subset C of T_i^s such that (i) $m_{n-1,i,s}(C) \in \{t^s_{A_{i,1}}, t^s_{A_{i,2}}, t^s_{B_i}\}$ and (ii) $m_{n-1,i,s}$ is a homeomorphism of $\mathrm{int}(C)$ onto its image. A *n-cylinder of* T_i^u is defined similarly. Hence, a n-cylinder of T_i^s (resp. of T_i^u) is the projection into T_i^s (resp. T_i^u) of a stable leaf (resp. unstable leaf) n-cylinder segment. We say that $m_{i,s}$ has *bounded geometry* in an atlas $\mathcal{B}'_i$, if there is $\kappa_1 > 0$ such that $\kappa_1^{-1} < |e(C_1)|/|e(C_2)| < \kappa_1$ for every n-cylinders C_1 and C_2, with a common endpoint, and for every chart (e, U) of the atlas $\mathcal{B}'_i$ with $C_1 \cup C_2 \subset U$.

Lemma 3.1. *Given a C^{1+} Anosov structure $\mathcal{S}_i$, the Markov maps $m_{i,s}$ are C^{1+} local diffeomorphisms and have bounded geometry with respect to the $C^{1+\alpha}$ atlases $\mathcal{B}'_s(\mathcal{S}_i)$ and $\mathcal{B}'_s(\mathcal{S}_{i-1})$ and the Markov maps $m_{i,u}$ are C^{1+} local diffeomorphisms and have bounded geometry with respect to the $C^{1+\alpha}$ atlases $\mathcal{B}'_u(\mathcal{S}_i)$ and $\mathcal{B}'_u(\mathcal{S}_{i+1})$.*

A *stable exchange map* $\tilde{\theta}_i : \pi_{\mathrm{T}_i^s}(I) \to \pi_{\mathrm{T}_i^s}(J)$ is the homeomorphism given by $\tilde{\theta}_i \circ \pi_{\mathrm{T}_i^s} = \pi_{\mathrm{T}_i^s} \circ \theta_i$, where $\theta_i : I \to J$ is a basic stable holonomy. The *stable exchange pseudo-group* $E_{i,s}$ is the set of all stable exchange maps. The *unstable exchange pseudo-group* $E_{i,u}$ is defined similarly.

Lemma 3.2. *The elements of the exchange pseudo-groups $E_{i,s}$ and $E_{i,u}$ are C^{1+} diffeomorphisms with respect to the C^{1+} structures $\mathcal{B}_s(\mathcal{S}_i)$ and $\mathcal{B}_u(\mathcal{S}_i)$, respectively.*

See the proof of Lemma 3.2 in [22].

Definition 3.3. A C^{1+} *stable self-renormalizable sequence* $\mathcal{BB}_s = (\mathcal{B}_0, \ldots, \mathcal{B}_{N-1})$ is a sequence of C^{1+} structures $\mathcal{B}_i$ with the following properties:

(i) The elements of the exchange pseudo-group $E_{i,s}$ are C^{1+} diffeomorphisms with respect to the C^{1+} structure $\mathcal{B}_i$.
(ii) For some $\alpha > 0$, the Markov maps $m_{i,s} : \mathrm{T}_i^s \to \mathrm{T}_{i-1}^s$ are $C^{1+\alpha}$ local diffeomorphisms and have bounded geometry with respect to some atlasses $\mathcal{B}'_i \subset \mathcal{B}_i$ and $\mathcal{B}'_{i-1} \subset \mathcal{B}_{i-1}$.

The C^{1+} *unstable self-renormalizable sequence* $\mathcal{BB}_u$ is defined similarly. Given a C^{1+} structure $\mathcal{S}$, define

$$\mathcal{BB}_s(\mathcal{S}) = (\mathcal{B}_s(\mathcal{S}_0), \ldots, \mathcal{B}_s(\mathcal{S}_{N-1}))$$

and

$$\mathcal{BB}_u(\mathcal{S}) = (\mathcal{B}_u(\mathcal{S}_0), \ldots, \mathcal{B}_u(\mathcal{S}_{N-1})).$$

Theorem 3.4. *The map $\mathcal{S} \mapsto (\mathcal{BB}_s(\mathcal{S}), \mathcal{BB}_u(\mathcal{S}))$ determines a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} Anosov diffeomorphisms and pairs of C^{1+} stable and unstable self-renormalizable sequences.*

See the proof of Theorem 3.4 in [27].

4. **Circle diffeomorphisms.** Let the *ATW partition* of $\mathbb{T}_i$ be given by $\mathcal{ATW}_i = \{A_i, B_i\}$, where $A_i = \pi_i([0,1] \times [0,1])$ and $B_i = \pi_i([-\gamma_{i-1}, 0] \times [0, \gamma_{i-2}])$. We form the *stable circle* $\mathbb{S}_i^s = t^s_{A_i} \bigsqcup t^s_{B_i} / \sim$. Let $\pi_{\mathbb{S}_i^s} : \bigsqcup_{R_i \in \mathcal{ATW}_i} R_i \to \mathbb{S}_i^s$ be the natural projection sending $x \in R_i$ to the point $\ell_i^u(x, R_i)$ in $\mathbb{S}_i^s$. Topologically, the space $\mathbb{S}_i = \mathbb{S}_i^s$ is a counterclockwise oriented circle.

Let $I_{\mathbb{S}}$ be an arc in $\mathbb{S}_i$ and I a leaf segment such that $\pi_{\mathbb{S}_i}(I) = I_{\mathbb{S}}$. The chart $i : I \to \mathbb{R}$ in $\mathcal{L}_s(\mathcal{S}_i)$ determines a *circle chart* $i_{\mathbb{S}} : I_{\mathbb{S}} \to \mathbb{R}$ for $I_{\mathbb{S}}$ given by $i_{\mathbb{S}} \circ \pi_{\mathbb{S}_i} = i$. We denote by $\mathcal{A}'_s(\mathcal{S}_i)$ the set of all circle charts $i_{\mathbb{S}}$ determined by the charts i in $\mathcal{L}_s(\mathcal{S}_i)$. Given any circle charts $i_{\mathbb{S}} : I_{\mathbb{S}} \to \mathbb{R}$ and $j_{\mathbb{S}} : J_{\mathbb{S}} \to \mathbb{R}$, the overlap map $j_{\mathbb{S}} \circ i_{\mathbb{S}}^{-1}$ is equal to $j \circ \theta_i \circ i^{-1}$, where $i = i_{\mathbb{S}} \circ \pi_{\mathbb{S}_G} : I \to \mathbb{R}$ and $j = j_{\mathbb{S}} \circ \pi_{\mathbb{S}_G} : J \to \mathbb{R}$ are charts in $\mathcal{L}_s(\mathcal{S}_i)$, and

$$\theta_i : i^{-1}(i_{\mathbb{S}}(I_{\mathbb{S}} \cap J_{\mathbb{S}})) \to j^{-1}(j_{\mathbb{S}}(I_{\mathbb{S}} \cap J_{\mathbb{S}}))$$

is a basic stable holonomy. By Theorem 2.1 in Pinto and Rand [26], there exists $\alpha > 0$ such that, for all circle charts $i_{\mathbb{S}}$ and $j_{\mathbb{S}}$ contained in $\mathcal{A}'_s(\mathcal{S}_i)$, the overlap maps $j_{\mathbb{S}} \circ i_{\mathbb{S}}^{-1} = j \circ \theta_i \circ i^{-1}$ are $C^{1+\alpha}$ diffeomorphisms with a uniform bound in the $C^{1+\alpha}$ norm. Hence, $\mathcal{A}'_s(\mathcal{S}_i)$ is a C^{1+} atlas. Let $\mathcal{A}_s(\mathcal{S}_i)$ be the C^{1+} structure that is C^{1+} compatible with the C^{1+} atlas $\mathcal{A}'_s(\mathcal{S}_i)$. Let $\mathcal{A}_s(\mathcal{L})$ be the C^{1+} structure C^{1+} compatible with the atlas $\mathcal{A}'_s$. The affine canonical structure $\underline{\mathcal{S}}_i$ determines an affine stable and unstable canonical structures $\mathcal{A}_s(\underline{\mathcal{S}}_i)$ and $\mathcal{A}_u(\underline{\mathcal{S}}_i)$, respectively.

Suppose that I and J are stable leaf segments and $\theta : I \to J$ is a holonomy map such that, for every $x \in I$, the unstable leaf segments with endpoints x and $\theta(x)$ cross once, and only once, a stable boundary of a pre-Markov rectangle. We define the *arc rotation map* $\tilde{\theta}_i : \pi_{\mathbb{S}_i}(I) \to \pi_{\mathbb{S}_i}(J)$, associated to θ_i, by $\tilde{\theta}_i(\pi_{\mathbb{S}_i}(x)) = \pi_{\mathbb{S}_i}(\theta_i(x))$. By Theorem 2.1 in Pinto and Rand [26], there exists $\alpha > 0$ such that the holonomy $\theta_i : I \to J$ is a $C^{1+\alpha}$ diffeomorphism, with respect to the C^{1+} lamination atlas $\mathcal{L}_s(\mathcal{S}_i)$. Hence, the arc rotation maps $\tilde{\theta}_i$ are C^{1+} diffeomorphisms with respect to the C^{1+} structure $\mathcal{A}_s(\mathcal{S}_i)$. Furthermore, the rotation maps $\tilde{\theta}_i$ are affine with respect to the affine canonical atlas $\mathcal{A}'_s(\underline{\mathcal{L}}_i)$.

Lemma 4.1. *There is a well-defined C^{1+} circle diffeomorphism $g_{s,i}; \mathbb{S}_i^s \to \mathbb{S}_i^s$ with respect to the C^{1+} structure $\mathcal{A}_s(\mathcal{S}_i)$, such that $g_{s,i}|_{\pi_{\mathbb{S}_i}(I)} = \tilde{\theta}_i$, for every arc rotation map $\tilde{\theta}_i$. Furthermore, $g_{s,i}$ has rotation number γ_i and $g_{s,i}$ is the rigid rotation with respect to the affine canonical structure $\mathcal{A}_s(\underline{\mathcal{S}}_i)$.*

See the proof of Lemma 4.1 in [22].

Let $\mathcal{A}^s_{0,*}$ be the C^{1+} structure on T^s_0 given by the pushforward of the C^{1+} structure $\mathcal{A}^s_0$ on $\mathbb{S}_0$ by the natural projection map $\pi_{\mathbb{S}_i, \mathrm{T}_i^s} : \mathbb{S}_i \to \mathrm{T}_i^s$.

Theorem 4.2. *The circle map $g_{s,0}$ is a C^{1+} diffeomorphism with respect to the C^{1+} structure $\mathcal{A}_0$ if, and only if, the elements of the stable exchange pseudo-group $E_{0,s}$ are C^{1+} diffeomorphisms with respect to the C^{1+} structure $\mathcal{A}^s_{0,*}$.*

See the proof of Theorem 4.2 in [22].

Let the C^{1+} structure $\mathcal{A}^s_{i,*}$ be the pullback of the C^{1+} structure $\mathcal{A}^s_{0,*}$ by the sequence of Markov maps $m_{i,0,s}$. The circle diffeomorphism g_0 is a C^{1+} periodic point of renormalization with respect to the C^{1+} structure $\mathcal{A}_0$ if, and only if, the sequence $(\mathcal{A}^s_{0,*}, \ldots, \mathcal{A}^s_{N-1,*})$ is a C^{1+} stable self-renormalizable sequence (see [3] and [22]).

A similar construction holds for the unstable foliation determining a C^{1+} circle diffeomorphism $g_{u,i} : \mathbb{S}^u_i \to \mathbb{S}^u_i$ with respect to the C^{1+} structure $\mathcal{A}_u(\mathcal{S}_i)$ with rotation number γ_{i-1}. Furthermore, the map $\mathcal{S} \mapsto (\mathcal{A}_s(\mathcal{S}), \mathcal{A}_u(\mathcal{S}))$ determines a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} Anosov diffeomorphisms and pairs of C^{1+} circle diffeomorphisms $(g_{s,i}, g_{u,i})$ that are C^{1+} periodic points of renormalization with respect to the C^{1+} structures $(\mathcal{A}_s(\mathcal{S}_i), \mathcal{A}_u(\mathcal{S}_i))$ (see [22]).

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