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LOCALIZATION AND PRICES IN THE QUADRATIC HOTELLING MODEL WITH UNCERTAINTY

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Abstract. For the quadratic Hotelling model, we study the optimal localization and price strategies under incomplete information on the production costs of the firms. We compute explicitly the pure Bayesian-Nash price duopoly equilibrium and we prove that it does not depend upon the distributions of the production costs of the firms, except on their first moments. We find when the maximal differentiation is a local optimum for the localization strategy of both firms.

1. **Introduction.** Since the seminal work of Hotelling [14], the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets (see [13, 16, 20, 19, 21, 22]). In his model, Hotelling presents a city represented by a line segment where a uniformly distributed continuum of consumers have to buy a homogeneous good. Consumers have to support linear transportation costs when buying the good in one of the two firms of the city. The firms compete in a two-staged location-price game, where simultaneously choose their location and afterwards set their prices in order to maximize their profits. Hotelling concluded that firms would agglomerate at the center of the line, an observation referred as the "Principle of Minimum Differentiation".

In 1979, D'Aspremont et al. [1] show that the "Principle of Minimum Differentiation" is invalid, since there was no price equilibrium solution for all possible locations of the firms, in particular when they are not far enough from each other. Moreover, in the same article, D'Aspremont et al.[1] introduce a modification in the Hotelling model, considering quadratic transportation costs instead of linear. The introduction of this feature removed the discontinuities verified in the profit and demand functions, which was a problem in Hotelling model and they show that, under quadratic transportation costs, a price equilibrium exists for all locations and a location equilibrium exists and involves maximum product differentiation, i.e. the firms opt to locate at the extremes of the line.

Hotelling and D'Aspremont et al. consider that the production costs of both firms are equal to zero. Ziss [23] introduced a modification in the model of D'Aspremont

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et al. [1] allowing for different production costs between the two firms and examines the effect of heterogeneous production technologies on the location problem. Ziss showed that a price equilibrium exists for all locations and concluded that when the difference between the production costs is small, a price and location equilibrium exists in which the firms prefer to be located at the opposite extremes of the line. However, if the difference between the production costs is sufficiently large, a location equilibrium does not exist.

Boyer et al. [4] and Biscaia and Sarmento [2] extended the work of Ziss considering that the uncertainty on the productions costs only during the first subgame in location strategies. Then the production costs are revealed to the firms before the firms have to choose their optimal price strategies and so the second subgame has complete information.

In this paper, we study the quadratic Hotelling model with incomplete information in the production costs of both firms. The incomplete information consists in each firm to know its production cost but to be uncertain about the competitor cost as usual in oligopoly theory (see [5, 6, 7, 8, 9, 10, 11, 12, 17, 18]). However, in contrast with Boyer et al. [4], the production costs are not revealed to the firms before the firms have to choose their price strategy. Furthermore, our results are universal, in the incomplete information scenario, in the sense that they apply to all probability distributions in the production costs.

We say that the Bayesian-Nash price strategy has the duopoly property if both firms have non-empty market for every pair of production costs. We introduce the bounded uncertain costs and location $BUCL1$ condition that defines a bound for the production costs in terms only of the exogenous variables that are the transportation cost; the road length of the segment line; and the localization of both firms (see section 6). We prove that there is a local optimum price strategy with the duopoly property if and only if the bounded uncertain costs and location $BUCL1$ condition holds. We compute explicitly the formula for the local optimum price strategy that is simple and leaves clear the influence of the relevance economic exogenous quantities in the price formation. In particular, we observe that the local optimum price strategy do not depend on the distributions of the production costs of the firms, except on their first moments.

We introduce two mild additional bounded uncertain costs and location $BUCL2$ and $BUCL3$ conditions. Under the $BUCL1$ and $BUCL2$ conditions, we prove that the local optimum price strategy is a Bayesian-Nash price strategy. Assuming that the firms choose the Bayesian-Nash price strategy, under the $BUCL3$ condition, we prove that the maximal differentiation is a local optimum for the localization strategy of both firms.

2. **Local optimum price strategy under complete information.**

The buyers of a commodity will be supposed uniformly distributed along a line with length l, where two firms A and B located at respective distances a and b from the endpoints of the line sell the same commodity with unitary *production costs* c_A and c_B. We assume without loss of generality that $a \geq 0$, $b \geq 0$ and $l - a - b \geq 0$. No customer has any preference for either seller except on the ground of price plus *transportation cost* t.

Denote A's *price* by p_A and B's *price* by p_B. The point of division $x = x(p_A, p_B) \in]0, l[$ between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or

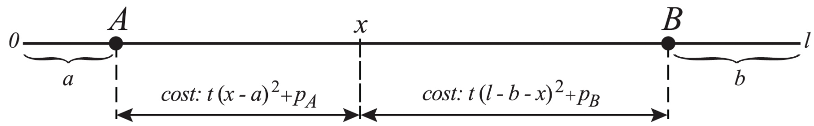

Figure 1. Hotelling's linear city with quadratic transportation costs

from B (see Figure 1). The point x is the location of the *indifferent consumer* to buy from firm A or firm B, if

$$p_A + t\,(x-a)^2 = p_B + t\,(l-b-x)^2$$

Let

$$m = l - a - b \quad \text{and} \quad \Delta_l = a - b.$$

Solving for x, we obtain

$$x = \frac{p_B - p_A}{2\,t\,m} + \frac{l + \Delta_l}{2}.$$

Both firms have a non-empty market share if, and only if, $x \in]0, l[$. Hence, the prices will have to satisfy

$$|p_A - p_B - t\,m\,\Delta_l| < t\,m\,l \tag{1}$$

Assuming inequality (1), both firms A and B have a non-empty demand (x and $l - x$) and the *profits* of the two firms are defined respectively by

$$\pi_A = (p_A - c_A)\,x = (p_A - c_A)\left(\frac{p_B - p_A}{2\,t\,m} + \frac{l + \Delta_l}{2}\right) \tag{2}$$

and

$$\pi_B = (p_B - c_B)\,(l - x) = (p_B - c_B)\left(\frac{p_A - p_B}{2\,t\,m} + \frac{l - \Delta_l}{2}\right). \tag{3}$$

Two of the fundamental economic quantities in oligopoly theory are the consumer surplus CS and the welfare W. The consumer surplus is the gain of the consumers community for given price strategies of both firms. The welfare is the gain of the state that includes the gains of the consumers community and the gains of the firms for given price strategies of both firms.

Let us denote by v_T the total amount that consumers are willing to pay for the commodity. The total amount $v(y)$ that a consumer located at y pays for the commodity is given by

$$v(y) = \begin{cases} p_A + t\,(y-a)^2 & \text{if } 0 < y < x; \\ p_B + t\,(l-b-y)^2 & \text{if } x < y < l. \end{cases}$$

The *consumer surplus* CS is the difference between the total amount that a consumer is willing to pay v_T and the total amount that the consumer pays $v(y)$

$$CS = \int_0^l v_T - v(y)\mathrm{d}y. \tag{4}$$

The *welfare* W is given by adding the profits of firms A and B with the consumer surplus

$$W = CS + \pi_A + \pi_B. \tag{5}$$

Definition 2.1. A price strategy $(\underline{p}_A, \underline{p}_B)$ for both firms is a *local optimum price strategy* if (i) for every small deviation of the price $\underline{p}_A$ the profit π_A of firm A decreases, and for every small deviation of the price $\underline{p}_B$ the profit π_B of firm B decreases (local optimum property); and (ii) the indifferent consumer exists, i.e. $0 < \underline{x} < l$ (duopoly property).

Let us compute the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$. Differentiating π_A with respect to p_A and π_B with respect to p_B and equalizing to zero, we obtain the first order conditions (FOC). The FOC imply that

$$\underline{p}_A = t\, m \left(l + \frac{\Delta_l}{3} \right) + \frac{1}{3} \left(2\, c_A + c_B \right) \tag{6}$$

and

$$\underline{p}_B = t\, m \left(l - \frac{\Delta_l}{3} \right) + \frac{1}{3} \left(c_A + 2\, c_B \right). \tag{7}$$

We note that the first order conditions refer to jointly optimizing the profit function (2) with respect to the price p_A and the profit function (3) with respect to the price p_B.

Since the profit functions (2) and (3) are concave, the second-order conditions for this maximization problem are satisfied and so the prices (6) and (7) are indeed maxima for the functions (2) and (3), respectively. The corresponding equilibrium profits are given by

$$\underline{\pi}_A = \frac{\left(m\, (3\, l + \Delta_l)\, t + c_B - c_A \right)^2}{18\, t\, m} \tag{8}$$

and

$$\underline{\pi}_B = \frac{\left(m\, (3\, l - \Delta_l)\, t + c_A - c_B \right)^2}{18\, t\, m}. \tag{9}$$

Furthermore, the indifferent consumer location corresponding to the maximizers $\underline{p}_A$ and $\underline{p}_B$ of the profit functions π_A and π_B is

$$\underline{x} = \frac{l}{2} + \frac{\Delta_l}{6} + \frac{c_B - c_A}{6\, t\, m}.$$

Finally, for the pair of prices $(\underline{p}_A, \underline{p}_B)$ to be a local optimum price strategy, we need assumption (1) to be satisfied with respect to these pair of prices. We observe that assumption (1) is satisfied with respect to the pair of prices $(\underline{p}_A, \underline{p}_B)$ if and only if the following condition with respect to the production costs is satisfied.

Definition 2.2. The Hotelling model satisfies the *bounded costs and location (BCL)* condition, if

$$|c_A - c_B - t\, m\, \Delta_l| < 3\, t\, m\, l.$$

We note that under the BCL condition the prices are higher than the production costs $\underline{p}_A > c_A$ and $\underline{p}_B > c_B$. Hence, there is a local optimum price strategy if and only if the BCL condition holds. Furthermore, under the BCL condition, the pair of prices $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy.

A strong restriction that the BCL condition imposes is that Δ_C converges to 0 when m tends to 0, i.e. when the differentiation in the localization tends to vanish.

Throughout this paper, consider

$$X_1 = v_T\, l - \frac{t}{3}\, l^3 + t\, l\, b\, (l - b) - t\, m\, l \left(l - \frac{\Delta_l}{3} \right)$$

and

$$X_2 = \frac{m\,t}{36}\,(45\,l^2 + 6\,l\,\Delta_l + 5\,\Delta_l^2).$$

By equation (4), the consumer surplus $\underline{CS}$ with respect to the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\begin{aligned}\underline{CS} &= \int_0^x v_T - \underline{p}_A - t\,(y-a)^2\,\mathrm{d}y + \int_x^l v - \underline{p}_B - t\,(l-b-y)^2\,\mathrm{d}y \\ &= v_T\,l + \underline{x}^2\,(l-a-b)\,t + (b\,(l-b)\,t - p_B)\,l - \frac{t}{3}\,l^3\end{aligned}$$

Hence,

$$\underline{CS} = X_1 - \frac{c_A + 2\,c_B}{3}\,l + \frac{(t\,m\,(3\,l + \Delta_l) + c_B - c_A)^2}{36\,t\,m}. \tag{10}$$

Adding (8), (9) and (10), we obtain the welfare

$$\underline{W} = X_1 - \frac{c_A + c_B}{2}\,l - \frac{5\,(c_A - c_B)}{18}\,\Delta_l + \frac{5\,(c_A - c_B)^2}{36\,t\,m} + X_2. \tag{11}$$

3. **Nash price strategy under complete information.** We note that, if a Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Nash equilibrium must also be global strategic maximum. In this section, we are going to show that this is the case.

Following D'Aspremont et al. [1], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals' price and supplying the whole market.

Let (p_A, p_B) be the local optimum price strategy. Firm A may gain the whole market, undercutting its rival by setting

$$p_A^M = \underline{p}_B - t\,m\,(l - \Delta_l).$$

In this case the profit amounts to

$$\pi_A^M = \frac{2}{3}\,(c_B - c_A + t\,m\,\Delta_l)\,l.$$

A similar argument is valid for store B. Undercutting this rival, setting

$$p_B^M = \underline{p}_A - t\,m\,(l + \Delta_l),$$

it would earn

$$\pi_B^M = \frac{2}{3}\,(c_A - c_B - t\,m\,\Delta_l)\,l.$$

The conditions for such undercutting not to be profitable are $\underline{\pi}_A \geq \pi_A^M$ and $\underline{\pi}_B \geq \pi_B^M$. Hence, proving that

$$\frac{(m\,(3\,l + \Delta_l)\,t + c_B - c_A)^2}{18\,t\,m} \geq \frac{2}{3}\,(t\,m\,\Delta_l - \Delta_C)\,l \tag{12}$$

is sufficient to prove that $\underline{\pi}_A \geq \pi_A^M$. Similarly, proving that

$$\frac{(m\,(3\,l - \Delta_l)\,t + c_A - c_B)^2}{18\,t\,m} \geq \frac{2}{3}\,(\Delta_C - t\,m\,\Delta_l)\,l \tag{13}$$

is sufficient to prove that $\underline{\pi}_B \geq \pi_B^M$.

However, conditions (12) and (13) are satisfied because they are equivalent to

$$(m\,(3\,l - \Delta_l)\,t + c_A - c_B)^2 \geq 0$$

and

$$(m\,(3\,l+\Delta_l)\,t+c_B-c_A)^2 \geq 0.$$

Therefore, if $(\underline{p}_A, \underline{p}_B)$ is a local optimum price strategy then $(\underline{p}_A, \underline{p}_B)$ is a Nash price equilibrium.

4. **Optimum localization equilibrium under complete information.** We are going to find when the maximal differentiation is a local optimum strategy assuming that the firms in second subgame choose the Nash price equilibrium strategy. For a complete discussion see Ziss [23].

We note that from (6) and (8), we can write the profit of firm A as

$$\underline{\pi}_A = \frac{(\underline{p}_A - c_A)^2}{2\,t\,(l-a-b)}.$$

Since

$$\frac{\partial \underline{p}_A}{\partial a} = -\frac{2}{3}\,t\,(l+a),$$

we obtain that

$$\frac{\partial \underline{\pi}_A}{\partial a} = -\frac{\underline{p}_A - c_A}{6\,t\,(l-a-b)^2}\,\left(c_A - c_B + t\,(l-a-b)\,(l+3\,a+b)\right).$$

Similarly, we obtain that

$$\frac{\partial \underline{\pi}_B}{\partial b} = \frac{\underline{p}_B - c_B}{6\,t\,(l-a-b)^2}\,\left(c_A - c_B - t\,(l-a-b)\,(l+a+3\,b)\right).$$

Therefore, the maximal differentiation $(a, b) = (0, 0)$ is a local optimum strategy if and only if

$$\frac{\partial \underline{\pi}_A}{\partial a}(0,0) = -\frac{\underline{p}_A - c_A}{6\,t\,l^2}\,\left(c_A - c_B + t\,l^2\right) < 0$$

and

$$\frac{\partial \underline{\pi}_B}{\partial b}(0,0) = \frac{\underline{p}_B - c_B}{6\,t\,l^2}\,\left(c_A - c_B - t\,l^2\right) < 0$$

Since

$$\frac{\underline{p}_A - c_A}{6\,t\,l^2} > 0 \quad \text{and} \quad \frac{\underline{p}_B - c_B}{6\,t\,l^2} > 0$$

the maximal differentiation $(a, b) = (0, 0)$ is a local optimum strategy if and only if

$$|c_A - c_B| < t\,l^2.$$

5. **Incomplete information on the production costs.** The incomplete information consists in each firm to know its production cost but to be uncertain about the competitor's cost. In this section, we introduce a simple notation that is fundamental for the elegance and understanding of the results presented in this paper.

Let the triples (I_A, Ω_A, q_A) and (I_B, Ω_B, q_B) represent (finite, countable or uncountable) sets of types I_A and I_B with σ-algebras Ω_A and Ω_B and probability measures q_A and q_B, over I_A and I_B, respectively.

We define the expected values $E_A(f)$, $E_B(f)$ and $E(f)$ with respect to the probability measures q_A and q_B as follows:

$$E_A(f) = \int_{I_A} f(z,w)\,\mathrm{d}q_A(z); \;\; E_B(f) = \int_{I_B} f(z,w)\,\mathrm{d}q_B(w)$$

and

$$E(f) = \int_{I_A}\int_{I_B} f(z,w)\,\mathrm{d}q_B(w)\mathrm{d}q_A(z).$$

Let $c_A : I_A \to \mathbb{R}_0^+$ and $c_B : I_B \to \mathbb{R}_0^+$ be measurable functions where $c_A^z = c_A(z)$ denotes the production cost of firm A when the type of firm A is $z \in I_A$ and $c_B^w = c_B(w)$ denotes the production cost of firm B when the type of firm B is $w \in I_B$. Furthermore, we assume that the expected values of c_A and c_B are finite

$$E(c_A) = E_A(c_A) = \int_{I_A} c_A^z \, \mathrm{d}q_A(z) < \infty;$$
$$E(c_B) = E_B(c_B) = \int_{I_B} c_B^w \, \mathrm{d}q_B(w) < \infty.$$

We assume that $\mathrm{d}q_A(z)$ denotes the probability of the *belief* of the firm B on the production costs of the firm A to be c_A^z. Similarly, we assume that $\mathrm{d}q_B(w)$ denotes the probability of the belief of the firm A on the production costs of the firm B to be c_B^w.

The simplicity of the following cost deviation formulas is crucial to express the main results of this paper in a clear and understandable way. The *cost deviations* of firm A and firm B

$$\Delta_A : I_A \to \mathbb{R}_0^+ \text{ and } \Delta_B : I_B \to \mathbb{R}_0^+$$

are given respectively by $\Delta_A(z) = c_A^z - E(c_A)$ and $\Delta_B(w) = c_B^w - E(c_B)$. The *cost deviation* between the firms

$$\Delta_C : I_A \times I_B \to \mathbb{R}_0^+$$

is given by $\Delta_C(z, w) = c_A^z - c_B^w$. Since the meaning is clear, we will use through the paper the following simplified notation:

$$\Delta_A = \Delta_A(z); \ \Delta_B = \Delta_B(w) \text{ and } \Delta_C = \Delta_C(z, w).$$

The *expected cost deviation* Δ_E between the firms is given by $\Delta_E = E(c_A) - E(c_B)$. Hence,

$$\Delta_C - \Delta_E = \Delta_A - \Delta_B.$$

Let V_A and V_B be the variances of the production costs c_A and c_B, respectively. We observe that

$$E(\Delta_C) = \Delta_E; \ E(\Delta_A^2) = E_A(\Delta_A^2) = V_A; \ E(\Delta_B^2) = E_B(\Delta_B^2) = V_B. \tag{14}$$

Furthermore,

$$E_A(\Delta_C^2) = \Delta_B^2 + V_A + \Delta_E\,(\Delta_E - 2\,\Delta_B); \tag{15}$$
$$E_B(\Delta_C^2) = \Delta_A^2 + V_B + \Delta_E\,(\Delta_E + 2\,\Delta_A); \tag{16}$$
$$E(\Delta_C^2) = \Delta_E^2 + V_A + V_B. \tag{17}$$

6. **Local optimal price strategy under incomplete information.** In this section, we introduce incomplete information in the classical Hotelling game and we find the local optimal price strategy. We introduce the bounded uncertain costs condition that allows us to find the local optimum price strategy.

A *price strategy* (p_A, p_B) is given by a pair of functions $p_A : I_A \to \mathbb{R}_0^+$ and $p_B : I_B \to \mathbb{R}_0^+$ where $p_A^z = p_A(z)$ denotes the price of firm A when the type of firm A is $z \in I_A$ and $p_B^w = p_B(w)$ denotes the price of firm B when the type of firm B is $w \in I_B$. We note that $E(p_A) = E_A(p_A)$ and $E(p_B) = E_B(p_B)$. The *indifferent consumer* $x : I_A \times I_B \to (0, l)$ is given by

$$x^{z,w} = \frac{p_B^w - p_A^z + t\,m\,(l + \Delta_l)}{2\,t\,m}. \tag{18}$$

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firm. Hence, it is the main economic information for both firms. However, the incomplete information prevents the firms to have access to their ex-post profits except after the firms have already decided their price strategies. The *ex-post profits* $\pi_A^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ and $\pi_B^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EP}(z,w) = \pi_A(z,w) = (p_A^z - c_A^z)\, x^{z,w}$$

and

$$\pi_B^{EP}(z,w) = \pi_B(z,w) = (p_B^w - c_B^w)\,(l - x^{z,w}).$$

The ex-ante profit of the firms is the expected profit of the firm that knows its production cost but are uncertain about the production cost of the competitor firm. The *ex-ante profits* $\pi_A^{EA} : I_A \to \mathbb{R}_0^+$ and $\pi_B^{EA} : I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EA}(z) = E_B(\pi_A^{EP}) \text{ and } \pi_B^{EA}(w) = E_A(\pi_B^{EP}). \tag{19}$$

We note that, the *expected profit* $E(\pi_A^{EP})$ of firm A is equal to $E_A(\pi_A^{EA})$ and the *expected profit* $E(\pi_B^{EP})$ of firm B is equal to $E_B(\pi_B^{EA})$.

The incomplete information forces the firms to have to choose their price strategies using their knowledge of their ex-ante profits, to which they have access, instead of the ex-post profits, to which they do not have access except after the price strategies are decided.

Definition 6.1. A price strategy $(\underline{p}_A, \underline{p}_B)$ for both firms is a *local optimum price strategy* if (i) for every $z \in I_A$ and for every small deviation of the price $\underline{p}_A^z$ the ex-ante profit $\pi_A^{EA}(z)$ of firm A decreases, and for every $w \in I_B$ and for every small deviation of the price $\underline{p}_B^w$ the ex-ante profit $\pi_B^{EA}(w)$ of firm B decreases (local optimum property); and (ii) for every $z \in I_A$ and $w \in I_B$ the indifferent consumer exists, i.e. $0 < \underline{x}^{z,w} < l$ (duopoly property).

We introduce the $BUCL1$ condition that has the crucial economical information that can be extracted from the exogenous variables. The $BUCL1$ condition allow us to know if there is, or not, a local optimum price strategy in the presence of uncertainty for the production costs of both firms.

Definition 6.2. The Hotelling model satisfies the *bounded uncertain costs and location ($BUCL1$)* condition 1, if

$$|\Delta_E - 3\,\Delta_C + 2\,\Delta_l\, t\, m| < 6\, t\, m\, l.$$

for all $z \in I_A$ and for all $w \in I_B$.

A strong restriction that the $BUCL1$ condition imposes is that Δ_c converges to 0 when m tends to 0, i.e. when the differentiation in the localization tends to vanish.

For $i \in \{A, B\}$, we define

$$c_i^m = \min_{z \in I_i}\{c_i^z\} \text{ and } c_i^M = \max_{z \in I_i}\{c_i^z\}.$$

Let

$$\overline{\Delta} = \max_{i,j \in \{A,B\}}\{c_i^M - c_j^m\}$$

Thus, the bounded uncertain costs and location $BUCL1$ is implied by the following stronger $SBUCL1$ condition.

Definition 6.3. The Hotelling model satisfies the *bounded uncertain costs and location ($SBUCL1$)* condition, if

$$\overline{\Delta} < t\, l\, m. \tag{20}$$

The following theorem is a key economical result in oligopoly theory. First, it tell us about the existence, or not, of a local optimum price strategy only by accessing a simple inequality in the exogenous variables and so available to both firms. Secondly, give us explicit and simple formulas that allow the firms to know the relevance of the exogenous variables in their price strategies and corresponding profits.

Theorem 6.4. *There is a local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ if and only if the $BUCL1$ condition holds. Under the $BUCL1$ condition, the expected prices of the local optimum price strategy are given by*

$$E(\underline{p}_A) = t\, m \left(l + \frac{\Delta_l}{3} \right) + E(c_A) - \frac{\Delta_E}{3}; \tag{21}$$

$$E(\underline{p}_B) = t\, m \left(l - \frac{\Delta_l}{3} \right) + E(c_B) + \frac{\Delta_E}{3}. \tag{22}$$

Furthermore, the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is unique and it is given by

$$\underline{p}_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \;\; \underline{p}_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}. \tag{23}$$

We observe that the difference between the expected prices of both firms has a very useful and clear economical interpretation in terms of the localization and expected cost deviations.

$$E(\underline{p}_A) - E(\underline{p}_B) = \frac{2\, t\, m\, \Delta_l + \Delta_E}{3}.$$

Furthermore, for different production costs, the differences between the optimal prices of a firm are proportional to the differences of the production costs

$$\underline{p}_A^{z_1} - \underline{p}_A^{z_2} = \frac{c_A^{z_1} - c_A^{z_2}}{2}.$$

and

$$\underline{p}_B^{w_1} - \underline{p}_B^{w_2} = \frac{c_B^{w_1} - c_B^{w_2}}{2}.$$

for all $z_1, z_2 \in I_A$ and $w_1, w_2 \in I_B$. Hence, half of the production costs value is incorporated in the price.

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firms. Hence it is the main economic information for both firms. By equation (23), the ex-post profit of firm A is

$$\underline{\pi}_A^{EP}(z, w) = \frac{(2\, t\, m\, (3\, l + \Delta_l) - 3\, \Delta_A - 2\, \Delta_E)\, (2\, t\, m\, (3\, l + \Delta_l) + \Delta_E - 3\, \Delta_C)}{72\, t\, m}$$

and the ex-post profit of firm B is

$$\underline{\pi}_B^{EP}(z, w) = \frac{(2\, t\, m\, (3\, l - \Delta_l) - 3\, \Delta_B + 2\, \Delta_E)\, (2\, t\, m\, (3\, l - \Delta_l) - \Delta_E + 3\, \Delta_C)}{72\, t\, m}.$$

The ex-ante profit of a firm is the expected profit of the firm that knows its production cost but is uncertain about the production costs of the competitor firm. Since $\underline{\pi}_A^{EP}(z,w)$ is given by

$$\frac{(2\,t\,m\,(3\,l+\Delta_l)-3\,\Delta_A-2\,\Delta_E)\,(2\,t\,m\,(3\,l+\Delta_l)+\Delta_E+3\,(c_B^w-c_A^z))}{72\,t\,m},$$

the ex-ante profit of firm A, $\underline{\pi}_A^{EA}(z)$, is

$$\frac{(2\,t\,m\,(3\,l+\Delta_l)-3\,\Delta_A-2\,\Delta_E)\,(2\,t\,m\,(3\,l+\Delta_l)+\Delta_E+3\,(E(c_B)-c_A^z))}{72\,t\,m}$$

Hence,

$$\underline{\pi}_A^{EA}(z)=\frac{(2\,t\,m\,(3\,l+\Delta_l)-3\,\Delta_A-2\,\Delta_E)^2}{72\,t\,m}. \tag{24}$$

Similarly, the ex-ante profit of firm B is

$$\underline{\pi}_B^{EA}(w)=\frac{(2\,t\,m\,(3\,l-\Delta_l)-3\,\Delta_B+2\,\Delta_E)^2}{72\,t\,m}. \tag{25}$$

Let α_A and α_B be given by

$$\alpha_A=\max\{E(c_B)-c_B^w : w\in I_B\} \quad\text{and}\quad \alpha_B=\max\{E(c_A)-c_A^z : z\in I_A\}.$$

The following corollary gives us the information of the market size of both firms by giving the explicit localization of the indifferent consumer with respect to the local optimum price strategy.

Corollary 1. *Under the BUCL1 condition, the indifferent consumer $x^{z,w}$ is given by*

$$\underline{x}^{z,w}=\frac{1}{2}\left(l+\frac{\Delta_l}{3}\right)+\frac{\Delta_E-3\,\Delta_C}{12\,t\,m}. \tag{26}$$

The pair of prices $(\underline{p}_A,\underline{p}_B)$ satisfies

$$\underline{p}_A^z-c_A^z\geq\alpha_A/2;\;\ \underline{p}_B^w-c_B^w\geq\alpha_B/2. \tag{27}$$

Proof of Theorem 6.4 and Corollary 1. Under incomplete information, each firm seeks to maximize its ex-ante profit. From (19), the ex-ante profit for firm A is given by

$$\begin{aligned}\pi_A^{EA}(z) &= \int_{I_B}(p_A^z-c_A^z)\left(\frac{p_B^w-p_A^z}{2\,t\,m}+\frac{l+\Delta_l}{2}\right)\mathrm{d}q_B(w)\\ &= (p_A^z-c_A^z)\left(\frac{E(p_B)-p_A^z}{2\,t\,m}+\frac{l+\Delta_l}{2}\right).\end{aligned} \tag{28}$$

From the first order condition FOC applied to the ex-ante profit of firm A we obtain

$$p_A^z=\frac{c_A^z+E(p_B)+t\,m\,(l+\Delta_l)}{2}. \tag{29}$$

Similarly,

$$\pi_B^{EA}(w)=(p_B^w-c_B^w)\left(\frac{E(p_A)-p_B^w}{2\,t\,m}+\frac{l-\Delta_l}{2}\right),$$

and, by the FOC, we obtain

$$p_B^w=\frac{c_B^w+E(p_A)+t\,m\,(l-\Delta_l)}{2}. \tag{30}$$

Then, from (29) and (30),

$$E(p_A) = \frac{E(c_A) + E(p_B) + t\,m\,(l + \Delta_l)}{2};$$
$$E(p_B) = \frac{E(c_B) + E(p_A) + t\,m\,(l - \Delta_l)}{2}.$$

Solving the system of two equations, we obtain that

$$E(p_A) = t\,m\left(l + \frac{\Delta_l}{3}\right) + \frac{E(c_B) + 2\,E(c_A)}{3};$$
$$E(p_B) = t\,m\left(l - \frac{\Delta_l}{3}\right) + \frac{E(c_A) + 2\,E(c_B)}{3}.$$

Hence, equalities (21) and (22) are satisfied. Replacing (22) in (29) and replacing (21) in (30) we obtain that

$$p_A^z = t\,m\left(l + \frac{\Delta_l}{3}\right) + \frac{c_A^z}{2} + \frac{E(c_A) + 2\,E(c_B)}{6};$$
$$p_B^w = t\,m\left(l - \frac{\Delta_l}{3}\right) + \frac{c_B^w}{2} + \frac{2\,E(c_A) + E(c_B)}{6}.$$

Hence, equation (23) is satisfied.

Replacing in equation (18) the values of $\underline{p}_A$ and $\underline{p}_B$ given by the equation (23) we obtain that the indifferent consumer $x^{z,w}$ is given by

$$x^{z,w} = \frac{1}{2}\left(l + \frac{\Delta_l}{3}\right) + \frac{3(c_B^w - c_A^z) + E(c_A) - E(c_B)}{12\,t\,m}.$$

Hence, equation (26) is satisfied. Therefore, $(\underline{p}_A, \underline{p}_B)$ satisfies property (ii) if and only if the $BUCL1$ condition holds.

Since the ex-ante profit functions (28) and (6) are concave, the second-order conditions for this maximization problem are satisfied and so the prices $\underline{p}_A^z$ and $\underline{p}_B^w$ are indeed maxima for the functions (28) and (6), respectively. Therefore, the pair $(\underline{p}_A^z, \underline{p}_B^w)$ satisfies property (i) and so $(\underline{p}_A^z, \underline{p}_B^w)$ is a local optimum price strategy.

Let us prove that $\underline{p}_A^z$ and $\underline{p}_B^w$ satisfy inequalities (27). By equation (23),

$$\underline{p}_A^z - c_A^z = t\,m\left(l + \frac{\Delta_l}{3}\right) - \frac{c_A^z}{2} + \frac{E(c_A) + 2\,E(c_B)}{6};$$
$$\underline{p}_B^w - c_B^w = t\,m\left(l - \frac{\Delta_l}{3}\right) - \frac{c_B^w}{2} + \frac{2\,E(c_A) + E(c_B)}{6}.$$

By the $BUCL1$ condition, for every $w \in I_B$, we obtain

$$\begin{aligned}
&6\left(\underline{p}_A^z - c_A^z - t\,m\left(l + \frac{\Delta_l}{3}\right)\right) = -3\,c_A^z + E(c_A) + 2\,E(c_B)\\
&= 3\,(E(c_B) - c_B^w) - 3\,(c_A^z - c_B^w) + E(c_A) - E(c_B)\\
&> 3\,(E(c_B) - c_B^w) - 6\,t\,l - 2\,\Delta_l\,t\,m.
\end{aligned}$$

Similarly, by the $BUCL1$ condition, for every $z \in I_A$, we obtain

$$\begin{aligned}
&6\left(\underline{p}_B^w - c_B^w - t\,m\left(l - \frac{\Delta_l}{3}\right)\right) = -3\,c_B^w + 2\,E(c_A) + E(c_B)\\
&= 3\,(E(c_A) - c_A^z) - 3\,(c_B^w - c_A^z) - E(c_A) + E(c_B)\\
&> 3\,(E(c_A) - c_A^z) - 6\,t\,l + 2\,\Delta_l\,t\,m.
\end{aligned}$$

Hence, inequalities (27) are satisfied. □

7. **Bayesian-Nash equilibrium.** We note that, if a Bayesian-Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Bayesian-Nash equilibrium must also be global strategic maximum. In this section, we are going to show that this is the case.

Following D'Aspremont et al. [1], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals's price and supplying the whole market for all admissible subsets of types I_A and I_B.

Definition 7.1. A price strategy $(\underline{p}_A, \underline{p}_B)$ for both firms is a *Bayesian-Nash*, if for every $z \in I_A$ and for every deviation of the price $\underline{p}_A^z$ the ex-ante profit $\pi_A^{EA}(z)$ of firm A decreases, and for every $w \in I_B$ and for every deviation of the price $\underline{p}_B^w$ the ex-ante profit $\pi_B^{EA}(w)$ of firm B decreases.

Let $(\underline{p}_A, \underline{p}_B)$ be the local optimum price strategy. Given the type w_0 of firm B, firm A may gain the whole market, undercutting its rival by setting

$$p_A^M(w_0) = \underline{p}_B^{w_0} - t\,m\,(l - \Delta_l) - \epsilon, \text{ with } \epsilon > 0.$$

Hence, by $BUCL1$ condition $p_A^M(w_0) \leq p_A^z$ for all $z \in I_A$. We observe that if firm A chooses the price $p_A^M(w_0)$ then by equalities (18) and (23) the whole market belongs to Firm A for all types w of firm B with $c^w \geq c^{w_0}$. Let

$$x(w; w_0) = \min\left\{l, \frac{p_B^w - p_A^M(w_0)}{2\,t\,m} + \frac{l + \Delta_l}{2}\right\}.$$

Thus, the *expected profit* with respect to the price $p_A^M(w_0)$ for firm A is

$$\pi_A^{EA,M}(w_0) = \int_{I_B} \left(p_A^M(w_0) - c_A^z\right) x(w; w_0)\, \mathrm{d}q_B(w).$$

Let $w_M \in I_B$ such that $c^{w_M} = c_B^M$. Since $c^{w_M} \geq c_B^{w_0}$ for every $w_0 \in I_B$, we obtain

$$\pi_A^{EA,M}(w_0) \leq \left(p_A^M(w_0) - c_A^z\right) l \leq \left(p_A^M(w_M) - c_A^z\right) l \tag{31}$$

Given the type z_0 of firm A, firm B may gain the whole market, undercutting its rival by setting

$$p_B^M(z_0) = \underline{p}_A^{z_0} - t\,m\,(l + \Delta_l) - \epsilon, \text{ with } \epsilon > 0.$$

Hence, by $BUCL1$ condition $p_B^M(z_0) \leq p_B^w$ for all $w \in I_B$. We observe that if firm B chooses the price $p_B^M(z_0)$ then by equalities (18) and (23) the whole market belongs to Firm B for all types z of firm A with $c^z \geq c^{z_0}$. Let

$$x(z; z_0) = \max\left\{0, \frac{p_B^M(z_0) - p_A^z}{2\,t\,m} + \frac{l + \Delta_l}{2}\right\}.$$

Thus, the *expected profit* with respect to the price $p_B^M(z_0)$ of firm B is

$$\pi_B^{EA,M}(z_0) = \int_{I_A} \left(p_B^M(z_0) - c_B^w\right) \left(l - x(z; z_0)\right) \mathrm{d}q_A(z).$$

Let $z_M \in I_A$ such that $c_A^{z_M} = c_A^M$. Since $c^{z_M} \geq c^{z_0}$ for every $z_0 \in I_A$, we obtain

$$\pi_B^{EA,M}(z_0) \leq \left(p_B^M(z_0) - c_B^w\right) l \leq \left(p_B^M(z_M) - c_B^w\right) l. \tag{32}$$

Remark 1. Under the $BUCL1$ condition, the strategic equilibrium $(\underline{p}_A, \underline{p}_B)$ is the unique pure Bayesian Nash equilibrium with the duopoly property if for every $z \in I_A$ and every $w \in I_B$,

$$\pi_A^{EA,M}(w) \leq \underline{\pi}_A^{EA}(z) \quad \text{and} \quad \pi_B^{EA,M}(z) \leq \underline{\pi}_B^{EA}(w). \tag{33}$$

Definition 7.2. The Hotelling model satisfies the *bounded uncertain costs and location (BUCL2)* condition, if

$$\begin{aligned}\Delta_E + 3\,\left(c_A^M + c_B^M - 2\,c_A^m\right) + \frac{\Delta_l\,(3\,c_A^M - E(c_A) - 2\,E(c_B))}{3\,l} \leq \\ \leq \frac{t\,m\,(3\,l - \Delta_l)^2}{3\,l} + \frac{\left(3\,c_A^M - E(c_A) - 2\,E(c_B)\right)^2}{12\,t\,m\,l}\end{aligned} \tag{34}$$

and

$$\begin{aligned}-\Delta_E + 3\,\left(c_A^M + c_B^M - 2\,c_B^m\right) - \frac{\Delta_l\,(3\,c_B^M - E(c_B) - 2\,E(c_A))}{3\,l} \leq \\ \leq \frac{t\,m\,(3\,l + \Delta_l)^2}{3\,l} + \frac{\left(3\,c_B^M - E(c_B) - 2\,E(c_A)\right)^2}{12\,t\,m\,l}.\end{aligned} \tag{35}$$

Thus, the bounded uncertain costs condition $BUCL2$ is implied by the following stronger $SBUCL2$ condition.

Definition 7.3. The Hotelling model satisfies the *strong bounded uncertain costs and location (SBUCL2)* condition, if

$$6\,\overline{\Delta} < l\,t\,m$$

We observe that the $SBUCL2$ condition implies $SBUCL1$ condition and so implies the $BUCL1$ condition.

Theorem 7.4. *If the Hotelling model satisfies the $BUCL1$ and $BUCL2$ conditions the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian Nash equilibrium.*

Corollary 2. *If the Hotelling model satisfies $SBUCL2$ condition the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian Nash equilibrium.*

Proof. By equalities (24) and (25), we obtain that $\underline{\pi}_A^{EA}(z_M) \leq \underline{\pi}_A^{EA}(z)$ and $\underline{\pi}_B^{EA}(w_M) \leq \underline{\pi}_B^{EA}(w)$ for all $z \in I_A$ and for all $w \in I_B$. Hence, putting conditions (31), (32) and (33) together, we obtain the following sufficient condition for the local optimal strategic prices $(\underline{p}_A, \underline{p}_B)$ to be a Bayesian Nash equilibrium:

$$(p_A^M(w_M) - c_A^m)\,l \leq \underline{\pi}_A^{EA}(z_M) \quad \text{and} \quad (p_B^M(z_M) - c_B^m)\,l \leq \underline{\pi}_B^{EA}(w_M). \tag{36}$$

By equalities (24) and (25) we obtain that

$$\underline{\pi}_A^{EA}(z_M) = \frac{(2\,t\,m\,(3\,l + \Delta_l) + E(c_A) + 2\,E(c_B) - 3\,c_A^M)^2}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w_M) = \frac{(2\,t\,m\,(3\,l - \Delta_l) + 2\,E(c_A) + E(c_B) - 3\,c_B^M)^2}{72\,t\,m}.$$

Also, from (23), we know that

$$\begin{aligned} p_A^M(w_M) - c_A^m &= \underline{p}_B^{w_M} - t\,m\,(l - \Delta_l) - \epsilon - c_A^m \\ &= \frac{1}{6}\,(4\,t\,m\,\Delta_l + 3\,c_B^M + 2\,E(c_A) + E(c_B) - 6\,c_A^m) - \epsilon. \end{aligned}$$

and

$$\begin{aligned} p_B^M(z_M) - c_B^m &= \underline{p}_A^{z_M} - t\,m\,(l+\Delta_l) - \epsilon - c_B^m \\ &= \frac{1}{6}\,(-4\,t\,m\,\Delta_l + 3\,c_A^M + E(c_A) + 2\,E(c_B) - 6\,c_B^m) - \epsilon. \end{aligned}$$

Hence, condition (36) holds if

$$\begin{aligned} &12\,t\,m\,l\,(4\,t\,m\,\Delta_l + 3\,c_B^M + 2\,E(c_A) + E(c_B) - 6\,c_A^m) \leq \\ &\leq (2\,t\,m\,(3\,l+\Delta_l) + E(c_A) + 2\,E(c_B) - 3\,c_A^M)^2 \end{aligned} \tag{37}$$

and

$$\begin{aligned} &12\,t\,m\,l\,(-4\,t\,m\,\Delta_l + 3\,c_A^M + E(c_A) + 2\,E(c_B) - 6\,c_B^m) \leq \\ &(2\,t\,m\,(3\,l-\Delta_l) + 2\,E(c_A) + E(c_B) - 3\,c_B^M)^2. \end{aligned} \tag{38}$$

Finally, we note that inequality (37) is equivalent to inequality (34) and that inequality (38) is equivalent to inequality (35). $\square$

8. **Optimum localization equilibrium under incomplete information.** We note that from (23) and (24), we can write the profit of firm A as

$$\underline{\pi}_A^{EA}(z) = \frac{(\underline{p}_A^z - c_A)^2}{2\,t\,(l-a-b)}.$$

Since

$$\frac{\partial \underline{p}_A^z}{\partial a} = -\frac{2}{3}\,t\,(l+a)$$

we have

$$\frac{\partial \underline{\pi}_A^{EA}}{\partial a} = \frac{\underline{p}_A - c_A}{12\,t\,(l-a-b)^2}\,\left(-2\,t\,(l-a-b)\,(l+3\,a+b) - 3\,\Delta_A - 2\,\Delta_E\right).$$

Similarly, we obtain that

$$\frac{\partial \underline{\pi}_B^{EA}}{\partial b} = \frac{\underline{p}_B - c_B}{12\,t\,(l-a-b)^2}\,\left(-2\,t\,(l-a-b)\,(l+3\,b+a) - 3\,\Delta_B + 2\,\Delta_E\right).$$

Therefore, the maximal differentiation $(a,b) = (0,0)$ is a local optimum strategy if and only if

$$\frac{\partial \underline{\pi}_A^{EA}}{\partial a}(0,0) = -\frac{\underline{p}_A - c_A}{12\,t\,l^2}\,\left(2\,t\,l^2 + 3\,\Delta_A + 2\,\Delta_E\right) < 0$$

and

$$\frac{\partial \underline{\pi}_B^{EA}}{\partial b}(0,0) = -\frac{\underline{p}_B - c_B}{12\,t\,l^2}\,\left(2\,t\,l^2 + 3\,\Delta_B - 2\,\Delta_E\right) < 0$$

Since

$$\frac{\underline{p}_A - c_A}{6\,t\,l^2} > 0 \quad \text{and} \quad \frac{\underline{p}_B - c_B}{6\,t\,l^2} > 0$$

the maximal differentiation $(a,b) = (0,0)$ is a local optimum strategy if and only if the following condition holds.

Definition 8.1. The Hotelling model satisfies the *bounded uncertain costs and location (BUCL3)* condition, if

$$2\,t\,l^2 + 3\,\Delta_A + 2\,\Delta_E > 0$$

for all $z \in I_A$ and

$$2\,t\,l^2 + 3\,\Delta_B - 2\,\Delta_E > 0$$

for all $w \in I_B$.

9. **Comparative profit analysis.** From now on, we assume that the $BUCL1$ condition holds and that the price strategy $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy determined in Theorem 6.4.

Let $\Delta_1 = \Delta_A + \Delta_B$ and $\Delta_2 = \Delta_A - \Delta_B$. We observe that the difference between the ex-post profits of both firms, $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_B^{EP}(z,w)$, has a very useful and clear economical interpretation in terms of the expected cost deviations and is given by

$$\frac{16\,t^2\,m^2\,l\,\Delta_l + 2\,t\,m\,(3\,l\,\Delta_2 - \Delta_l\,\Delta_1) + (\Delta_E - 3\,\Delta_C)\,(8\,t\,l\,m - \Delta_1)}{24\,t\,m}.$$

Furthermore, for different production costs, the differences between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1,w) - \underline{\pi}_A^{EP}(z_2,w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) - \Delta_E + 3\,(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, similarly, $\underline{\pi}_B^{EP}(z,w_1) - \underline{\pi}_B^{EP}(z,w_2)$ is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + \Delta_E + 3\,(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

We observe that the difference between the ex-ante profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4\,t\,m\,l - \Delta_1)\,(4\,(t\,m\,\Delta_l - \Delta_E) - 3\,\Delta_2)}{24\,t\,m}.$$

Furthermore, for different production costs, the differences between the ex-ante profit of firm A, $\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) - 4\,\Delta_E + 3\,(2\,E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, similarly, $\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2)$ is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 4\,\Delta_E + 3\,(2\,E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The difference between the ex-post and the ex-ante profit for a firm is the real deviation from the realized gain of the firm and the expected gain of the firm knowing its own production cost but being uncertain about the production cost of the other firm. It is the best measure of the risk involved for the firm given the uncertainty in the production costs of the other firm. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{24\,t\,m}\,(2\,t\,m\,(3\,l + \Delta_l) - 2\,\Delta_E - 3\,\Delta_A).$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{24\,t\,m}\,(2\,t\,m\,(3\,l - \Delta_l) + 2\,\Delta_E - 3\,\Delta_B).$$

Definition 9.1. The Hotelling model satisfies the *A-bounded uncertain costs and location $(A - BUCL)$* condition, if for all $z \in I_A$

$$3\,\Delta_A + 2\,\Delta_E < 2\,t\,m\,(3\,l + \Delta_l).$$

The Hotelling model satisfies the *B-bounded uncertain costs and location $(B - BUCL)$* condition, if for all $w \in I_B$

$$3\,\Delta_B - 2\,\Delta_E < 2\,t\,m\,(3\,l - \Delta_l).$$

The following corollary tells us that the sign of the risk of a firm has the opposite sign of the deviation of the competitor firm realized production cost from its average. Hence, under incomplete information the sign of the risk of a firm is not accessible to the firm. However, the probability of the sign of the risk of a firm to be positive or negative is accessible to the firm.

Corollary 3. *Under the A-bounded uncertain costs ($A-BUCL$) condition,*

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0. \tag{39}$$

Under the B-bounded uncertain costs ($B-BUCL$) condition,

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0. \tag{40}$$

The proof of the above corollary follows from a simple manipulation of the previous formulas for the ex-post and ex-ante profits.

The expected profit of the firm is the expected gain of the firm. We observe that the ex-ante and the ex-posts profits of both firms are strictly positive with respect to the local optimum price strategy. Hence, the expected profits of both firms are also strictly positive. Since the ex-ante profit $\underline{\pi}_A^{EA}(z)$ of firm A is equal to

$$\underline{\pi}_A^{EA}(z) = \frac{9\,\Delta_A^2 - 12\,\Delta_A\,(t\,m\,(3\,l+\Delta_l) - \Delta_E) + 4\,(t\,m\,(3\,l+\Delta_l) - \Delta_E)^2}{72\,t\,m},$$

from (14), we obtain that the expected profit of firm A is given by

$$E(\underline{\pi}_A^{EP}) = \frac{(t\,m\,(3\,l+\Delta_l) - \Delta_E)^2}{18\,t\,m} + \frac{V_A}{8\,t\,m}.$$

Similarly, the expected profit of firm B is given by

$$E(\underline{\pi}_B^{EP}) = \frac{(t\,m\,(3\,l-\Delta_l) + \Delta_E)^2}{18\,t\,m} + \frac{V_B}{8\,t\,m}.$$

The difference between the ex-ante and the expected profit of a firm is the deviation from the expected realized gain of the firm given the realization of its own production cost and the expected gain in average for different realizations of its own production cost, but being in both cases uncertain about the production costs of the competitor firm. It is the best measure of the quality of its realized production cost in terms of the expected profit over its own production costs.

Corollary 4. *The difference between the ex-ante profit and the expected profit for firm A is*

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,t\,m\,(3\,l+\Delta_l) - 3\,\Delta_A - 4\,\Delta_E) + 3\,V_A}{24\,t\,m}. \tag{41}$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,t\,m\,(3\,l-\Delta_l) - 3\,\Delta_B + 4\,\Delta_E) + 3\,V_B}{24\,t\,m}. \tag{42}$$

Proof. Let $Z = 2\,t\,m\,(3\,l+\Delta_l) - 2\,\Delta_E$. Hence,

$$\begin{aligned} E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) &= \frac{Z^2 - (Z - 3\,\Delta_A)^2}{72\,t\,m} + \frac{V_A}{8\,t\,m} \\ &= \frac{\Delta_A\,(2\,Z - 3\,\Delta_A) + 3\,V_A}{24\,t\,m}. \end{aligned}$$

and so equality (41) holds. The proof of equality (42) follows similarly. ∎

10. **Comparative consumer surplus and welfare analysis.** Consider throughout this section that $X = t\,m\,(3\,l + \Delta_l)$.

The ex-post consumer surplus is the realized gain of the consumers community for given outcomes of the production costs of both firms. Under incomplete information, by equation (4), the ex-post consumer surplus is

$$\underline{CS}^{EP} = X_1 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l - \frac{\Delta_B}{2}\,l + \frac{(2\,t\,m\,(3\,l+\Delta_l) + \Delta_E - 3\,\Delta_C)^2}{144\,t\,m}.$$

The expected value of the consumer surplus is the expected gain of the consumers community for all possible outcomes of the production costs of both firms. The expected value of the consumer surplus $E(\underline{CS}^{EP})$ is given by

$$\begin{aligned} E(\underline{CS}^{EP}) &= \int_{I_B}\int_{I_A} \underline{CS}^{EP}\,\mathrm{d}q_A(z)\,\mathrm{d}q_B(w) \\ &= X_1 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l + \frac{4\,(t\,m\,(3\,l+\Delta_l) - \Delta_E)^2 + 9\,(V_A + V_B)}{144\,t\,m}. \end{aligned}$$

We note that, from equalities (14) and (17), the expected value of

$$\frac{(2\,t\,m\,(3\,l+\Delta_l) + \Delta_E - 3\,\Delta_C)^2}{144\,t\,m}$$

is given by

$$\begin{aligned} &\frac{(2\,X + \Delta_E)^2 - 6\,E(\Delta_C)\,(2\,X+\Delta_E) + 9\,E(\Delta_C^2)}{144\,t\,m} \\ &= \frac{(2\,X + \Delta_E)^2 - 6\,\Delta_E\,(2\,X+\Delta_E) + 9\,(V_A + V_B + \Delta_E^2)}{144\,t\,m} \\ &= \frac{4\,(X - \Delta_E)^2 + 9\,(V_A + V_B)}{144\,t\,m}. \end{aligned}$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus measures the difference between the gain of the consumers for the realized outcomes of the production costs of both firms and the expected gain of the consumers for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the consumers for different outcomes of the production costs of both firms.

Corollary 5. *The difference between the ex-post consumer surplus and the expected value of the consumer surplus, $CS^{EP} - E(CS^{EP})$, is*

$$-\frac{\Delta_A + \Delta_B}{4}\,l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m}.$$

Proof.

$$\begin{aligned} &\underline{CS}^{EP} - E(\underline{CS}^{EP}) = \\ &= -\frac{\Delta_B}{2}\,l + \frac{(2\,X + \Delta_E - 3\,\Delta_C)^2 - 4\,(X - \Delta_E)^2 - 9\,(V_A + V_B)}{144\,t\,m} \\ &= -\frac{\Delta_B}{2}\,l + \frac{12\,X\,(\Delta_E - \Delta_C) + (\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m} \\ &= \frac{\Delta_E - \Delta_C - 2\,\Delta_B}{4}\,l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m} \end{aligned}$$

$$= -\frac{\Delta_A + \Delta_B}{4}\, l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\, t\, m}$$

$\square$

The ex-post welfare is the realized gain of the state that includes the gains of the consumers community and the gains of the firms for a given outcomes of the production costs of both firms. By equation (5), the ex-post welfare is

$$\begin{aligned} \underline{W}^{EP} &= \frac{5(\Delta_E - 3\Delta_C) + 3(\Delta_A - \Delta_B)}{36}\,\Delta_l - \frac{\Delta_A + \Delta_B + E(c_A) + E(c_B)}{2} l + \\ &+ X_1 + X_2 + X_3, \end{aligned}$$

where

$$X_3 = \frac{(3\,\Delta_C - \Delta_E)\,(9\,\Delta_C + \Delta_E)}{144\, t\, m}.$$

The expected value of the welfare is the expected gain of the state for all possible outcomes of the production costs of both firms. The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$\begin{aligned} E(\underline{W}^{EP}) &= \int_{I_B}\int_{I_A} \underline{W}^{EP} \mathrm{d}q_A(z)\, \mathrm{d}q_B(w) \\ &= X_1 + X_2 - \frac{E(c_A) + E(c_B)}{2}\, l - \frac{5\,\Delta_E}{18}\,\Delta_l + U_2 \end{aligned}$$

where

$$U_2 = \frac{20\,\Delta_E^2 + 27\,(V_A + V_B)}{144\, t\, m}.$$

We note that, from equalities (14) and (17), the expected value of X_3 is given by

$$\begin{aligned} U_2 &= \frac{27\, E(\Delta_C^2) - 6\, E(\Delta_C)\,\Delta_E - \Delta_E^2}{144\, t\, m} \\ &= \frac{27\,(\Delta_E^2 + V_A + V_B) - 7\,\Delta_E^2}{144\, t\, m} \\ &= \frac{20\,\Delta_E^2 + 27\,(V_A + V_B)}{144\, t\, m}. \end{aligned}$$

The difference between the ex-post welfare and the expected value of the welfare measures the difference in the gains of the state between the realized outcomes of the production costs of both firms and the expected gain of the state for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the state for different outcomes of the production costs of both firms. The difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = \frac{\Delta_A + \Delta_B}{2}\, l + \frac{\Delta_B - \Delta_A)}{3}\,\Delta_l + X_4$$

where

$$X_4 = \frac{9\,(\Delta_C^2 - V_A - V_B) - 2\,\Delta_C\,\Delta_E - 7\,\Delta_E^2}{48\, t\, m}.$$

11. **Example: Symmetric Hotelling.** A Hotelling game is *symmetric*, if $(I_A, \Omega_A, q_A) = (I_B, \Omega_B, q_B)$ and $c = c_A = c_B$. Hence, we observe that all the formulas of this section hold with the following simplifications

$$\Delta_E = 0;\;\; E(c) = E(c_A) = E(c_B) \text{ and } V = V_A = V_B.$$

The bounded uncertain costs in the symmetric case can be written in the following simple way.

Definition 11.1. The symmetric Hotelling model satisfies the *bounded uncertain costs (BUCL1)* condition, if

$$|2\,\Delta_l\,t\,m - 3\,\Delta_C| < 6\,t\,m\,l.$$

for all $z \in I_A$ and for all $w \in I_B$.

The Hotelling model with incomplete symmetric information satisfies the *bounded uncertain costs (BUCL2)* condition, if

$$6\,(c^M - c^m) + \frac{\Delta_l\,(c^M - E(c))}{l} \le \frac{t\,m\,(3\,l - \Delta_l)^2}{3\,l} + \frac{3\,(c^M - E(c))^2}{4\,t\,m\,l}$$

and

$$6\,(c^M - c^m) - \frac{\Delta_l\,(c^M - E(c))}{l} \le \frac{t\,m\,(3\,l + \Delta_l)^2}{3\,l} + \frac{3\,(c^M - E(c))^2}{4\,t\,m\,l}.$$

Under the $BUC1$ condition, the expected prices of the local optimum price strategy have the simple expression

$$E(\underline{p}_A) = t\,m\left(l + \frac{\Delta_l}{3}\right) + E(c); E(\underline{p}_B) = t\,m\left(l - \frac{\Delta_l}{3}\right) + E(c)$$

By Proposition 6.4, for the Hotelling game with incomplete symmetric information, the local optimum price strategy (p_A, p_B) has the form

$$p_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2};\; p_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}.$$

The ex-post profit of firm A and firm B are, respectively

$$\underline{\pi}_A^{EP}(z, w) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A)\,(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_C)}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EP}(z, w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B)\,(2\,t\,m\,(3\,l - \Delta_l) + 3\,\Delta_C)}{72\,t\,m}.$$

The difference between the ex-post profits, $\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_B^{EP}(z, w)$, of both firms is given by

$$\frac{16\,t^2\,m^2\,l\,\Delta_l + 2\,t\,m\,(3\,l\,\Delta_C - \Delta_l\,(\Delta_A + \Delta_B)) - 3\,\Delta_C\,(8\,t\,l\,m - \Delta_A - \Delta_B)}{24\,t\,m}.$$

Furthermore, for different production costs, the difference between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1, w) - \underline{\pi}_A^{EP}(z_2, w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) + 3\,(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, for different production costs, the difference between the ex-post profit of firm B, $\underline{\pi}_B^{EP}(z, w_1) - \underline{\pi}_B^{EP}(z, w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 3\,(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The ex-ante profit of firm A and of firm B are, respectively

$$\underline{\pi}_A^{EA}(z) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A)^2}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B)^2}{72\,t\,m}.$$

The difference between the ex-ante profits of both firms is given by

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4\,t\,m\,l - \Delta_A - \Delta_B)\,(4\,t\,m\,\Delta_l - 3\,\Delta_C)}{24\,t\,m}$$

Furthermore, for different production costs, the differences between the ex-ante profits of a firm are given by

$$\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2) = \frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) + 3\,(2\,E(c) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2) = \frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 3\,(2\,E(c) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{24\,t\,m}\,\left(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A\right).$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{24\,t\,m}\,\left(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B\right).$$

We observe that that the $A-BUCL$ and $B-BUCL$ conditions are implied by the $BUCL1$ condition. Hence, Corollary 3 can be rewritten without any restriction, i.e.

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \;\text{ if and only if }\; \Delta_B < 0;$$

and

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \;\text{ if and only if }\; \Delta_A < 0.$$

The expected profit of firm A and firm B are

$$E(\underline{\pi}_A^{EP}) = \frac{t\,m\,(3\,l + \Delta_l)^2}{18} + \frac{V}{8\,t\,m}$$

and

$$E(\underline{\pi}_B^{EP}) = \frac{t\,m\,(3\,l - \Delta_l)^2}{18} + \frac{V}{8\,t\,m}.$$

The difference between the ex-ante profit and the expected profit for firm A is

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A) + 3\,V}{24\,t\,m}.$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B) + 3\,V}{24\,t\,m}.$$

The ex-post consumer surplus is

$$\underline{CS}^{EP} = X_1 - E(c)\,l - \frac{\Delta_B}{2}\,l + \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_C)^2}{144\,t\,m}.$$

The expected value of the consumer surplus is

$$E(\underline{CS}^{EP}) = X_1 - E(c)\,l + \frac{4\,t^2\,m^2\,(3\,l + \Delta_l)^2 + 18\,V}{144\,t\,m}.$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus is

$$\underline{CS}^{EP} - E(\underline{CS}^{EP}) = -\frac{\Delta_A + \Delta_B}{4}\,l - \frac{\Delta_C}{12}\,\Delta_l + \frac{9\,\Delta_C^2 - 18\,V}{144\,t\,m}$$

The ex-post welfare is

$$\underline{W}^{EP} = X_1 + X_2 - E(c)\, l - \frac{\Delta_C}{3}\, \Delta_l + \frac{27\, \Delta_C^2}{144\, t\, m},$$

The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$E(\underline{W}^{EP}) = X_1 + X_2 - E(c)\, l - \frac{5\, \Delta_E}{18}\, \Delta_l + \frac{27\, (V_A + V_B)}{144\, t\, m}.$$

The difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = \frac{\Delta_A + \Delta_B}{2}\, l - \frac{\Delta_C)}{3}\, \Delta_l + \frac{9\, (\Delta_C^2 - 2\, V)}{48\, t\, m}.$$

12. **Conclusion.** We proved that there is a local optimum price strategy with the duopoly property if and only if the bounded uncertain costs and location $BUCL1$ condition holds. The explicit formulas of the local optimum price strategy determine prices for both firms that are affine with respect to the expected costs of both firms and to its own costs. Under the $BUCL1$ and $BUCL2$ conditions, we proved that the local optimum price strategy is a Bayesian-Nash price strategy. Assuming that the firms choose the Bayesian-Nash price strategy, under the $BUCL3$ condition, we proved that the maximal differentiation is a local optimum for the localization strategy of both firms.

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