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Anosov Diffeomorphisms and γ-Tilings

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Abstract: We consider a toral Anosov automorphism $G_\gamma : \mathbb{T}_\gamma \to \mathbb{T}_\gamma$ given by $G_\gamma(x, y) = (ax + y, x)$ in the $< v, w >$ base, where $a \in \mathbb{N}\backslash\{1\}$, $\gamma = 1/(a + 1/(a + 1/\ldots))$, $v = (\gamma, 1)$ and $w = (-1, \gamma)$ in the canonical base of $\mathbb{R}^2$ and $\mathbb{T}_\gamma = \mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$. We introduce the notion of γ-tilings to prove the existence of a one-to-one correspondence between (i) marked smooth conjugacy classes of Anosov diffeomorphisms, with invariant measures absolutely continuous with respect to the Lebesgue measure, that are in the isotopy class of G_γ; (ii) affine classes of γ-tilings; and (iii) γ-solenoid functions. Solenoid functions provide a parametrization of the infinite dimensional space of the mathematical objects described in these equivalences.

1. Introduction

This paper studies C^{1+} smooth conjugacy classes of Anosov diffeomorphisms G of the two-dimensional torus with a prescribed isotopy class G_γ. The ispotopy classes treated here belong to a countable family, each of which is determined by a Fibonacci like matrix. Inspired in the works of Jiang [14] and Pinto and Sullivan [30], Pinto et al. [21] introduced the notion of *golden tiling* and proved the one-to-one correspondences claimed in the abstract of this paper for the usual Fibonacci matrix, i.e. $a = 1$. Here, we extend their results for the case of Fibonacci like matrices, i.e. $a \in \mathbb{N}\backslash\{1\}$, by introducing the γ-tilings. Like the golden tilings, the γ-tilings record the infinitesimal geometric structure along the unstable leafs that are invariant under the action of G. The properties of the γ-tilings are given in a canonical way using the *γ-Fibonacci decomposition* of the natural numbers by the greedy algorithm (see [8]). The main contribution of this work consists in understanding the way this γ-Fibonacci decomposition encodes the combinatorics determined by the Markov partition of G along the unstable leafs that are invariant under the action of G (see Theorem 2).

In Theorem 3 it is exhibited a natural correspondence between (i) marked smooth conjugacy classes of Anosov diffeomorphisms, with invariant measures absolutely continuous with respect to Lebesgue, that are in the isotopy class of G_γ; (ii) γ-tilings; and (iii) γ-solenoid functions. Hence, Theorem 3 gives a characterization of marked smooth conjugacy classes of Anosov diffeomorphisms in terms of certain affine classes of tilings of the half-line by intervals. The tilings are essentially given by the tilings of the unstable manifold of a fixed point, obtained by considering its intersection with the rectangles in a canonical Markov partition. Theorem 4 is about rigidity: an Anosov diffeomorphism, with an invariant measure absolutely continuous with respect to Lebesgue, whose unstable holonomies are of class $C^{1+zigmund}$, is necessarily conjugate to the affine representative. The proofs of these results follow closely those in [21]. One of the main ingredients in the proof of Theorem 3 is the solenoid function of an Anosov map (see [26,28]), which is similar to the scaling function of a hyperbolic Cantor set (see [7,33]). The ratios between the lengths of consecutive intervals in an admissible tiling are precisely the values of the solenoid function in a certain sequence of points. Since this sequence of points is dense in the domain of the solenoid function, the main problem is to determine the conditions for a sequence of values on this set to extend to a Hölder continuous solenoid function. This leads naturally to the matching, the boundary, and the exponentially fast repetitive conditions that give the definition of admissible tilings (see Sect. 4.4). Section 3 contains most of the new combinatorial results needed to establish these conditions. Once Theorem 3 is established, Theorem 4 follows from known arguments.

As opposed to the case treated in [21], the Anosov automorphism G_γ considered in this paper has $a \geq 2$ fixed points instead of a single one. These fixed points are dynamically indistinguishable from each other: for each pair of these fixed points there is an orientation preserving conjugacy mapping one of the fixed points to the other. An unstable leaf is invariant under the dynamics if, and only if, the leaf passes through a fixed point. Hence, there are a of such invariant leafs. We associate to each one of these leafs the corresponding γ-tiling. Hence, each Anosov diffeomorphism determines a set of γ-tilings with cardinality a. Hence, two Anosov diffeomorphisms, with invariant measures absolutely continuous with respect to Lebesgue, are smooth conjugate if, and only if, they determine the same sets of γ-tilings. In fact, if two of these Anosov diffeomorphisms determine a same γ-tiling, then they determine the same sets of γ-tilings. Furthermore, we observe that a solenoid function determines a ratio function (see [26,28]) that measures the ratio of the asymptotic lengths of any pair of leaves with a common endpoint. Hence, any two γ-tilings are obtained from the same Anosov diffeomorphism, but using unstable lines passing through different fixed points, if, and only if, they determine the same ratio function.

Every Anosov automorphism determines a rigid rotation that is a periodic orbit of a renormalization operator (see, for instance, [1,28]). Putting together the results in this paper and in Pinto et al. [21], all Anosov automorphisms determining rigid rotations that are fixed points of a renormalization operator are treated. An open problem consists in extending these results to the isotopy classes of Anosov automorphisms on surfaces whose rigid rotation is not a fixed point of a renormalization operator. Another open problem consists in understanding the tilings determined by the marking of the stable manifolds (without reversing the Markov partition and time).

2. Anosov Diffeomorphisms and Circle Rotations

In this section, for clarity of exposition, we introduce the well-known Anosov automorphisms, marked Anosov diffeomorphisms and rigid rotations.

We start by fixing an integer $a \in \mathbb{N}\backslash\{1\}$ and consider the positive real number

$$\gamma = (-a + \sqrt{a^2+4})/2 = 1/(a + 1/(a + 1/\ldots)).$$

We observe that γ satisfies the key relation $a\gamma+\gamma^2 = 1$. Let $v = (\gamma, 1)$ and $w = (-1, \gamma)$ in the canonical base of $\mathbb{R}^2$. Let $\mathbb{T}_\gamma = \mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$ be the quotient space with the quotient topology. We consider the Anosov automorphism $G_\gamma : \mathbb{T}_\gamma \to \mathbb{T}_\gamma$ given by $G_\gamma(x, y) = (ax + y, x)$ in the base $< v, w >$.

The eigenvalues of G_γ are $\mu^- = -\gamma$ and $\mu^+ = 1/\gamma$. Let $\pi_\gamma : \mathbb{R}^2 \to \mathbb{T}_\gamma$ be the natural projection of $\mathbb{R}^2$ in $\mathbb{T}_\gamma$. Let $\tilde{A} = [0, 1] \times [0, 1]$ and $\tilde{B} = [-\gamma, 0] \times [0, \gamma]$ be rectangles in the plane $\mathbb{R}^2$. Let $A = \pi_\gamma(\tilde{A})$ and $B = \pi_\gamma(\tilde{B})$ be the projections of $\tilde{A}$ and $\tilde{B}$ in the torus $\mathbb{T}_\gamma$ (see Fig. 1). The *generating Markov partition* $\mathcal{M}_{G_\gamma}$ of G_γ is given by $\mathcal{M}_{G_\gamma} = \{A, B\}$. The stable and unstable manifolds of G_γ are the projection by π_γ of the horizontal and vertical lines of the plane, respectively.

2.1. Marked Anosov diffeomorphisms. Let $\mathcal{G}$ be the set of all pairs (G, z) with the property that (i) for some $\alpha > 0$, $G : \mathbb{T} \to \mathbb{T}$ is a $C^{1+\alpha}$ Anosov diffeomorphism in a two dimensional torus $\mathbb{T}$ with an invariant measure absolutely continuous with respect to Lebesgue measure; and (ii) z is a fixed point of G. Hence, the tangent bundle of G has a $C^{1+\alpha}$ uniformly hyperbolic splitting into a stable direction and an unstable direction and the holonomies are $C^{1+\alpha_1}$ smooth, for some $0 < \alpha_1 \le \alpha$ (see [19,28,32]). We call $(G, z) \in \mathcal{G}$ a *marked* C^{1+} Anosov diffeomorphism, because we have marked one of its fixed points. Let $\mathcal{G}_\gamma$ be the set of all pairs $(G, z) \in \mathcal{G}$ such that there is a topological conjugacy $h_G : \mathbb{T}_\gamma \to \mathbb{T}$ between the Anosov automorphism G_γ and the Anosov diffeomorphism G such that $z = h_G(\pi_\gamma(0, 0))$. We note that the topological conjugacy h_G, preserving the orientation, is uniquely determined. The generating Markov partition $\mathcal{M}(G, z)$ of G is formed by the rectangles $h_G(A)$ and $h_G(B)$, where A and B are

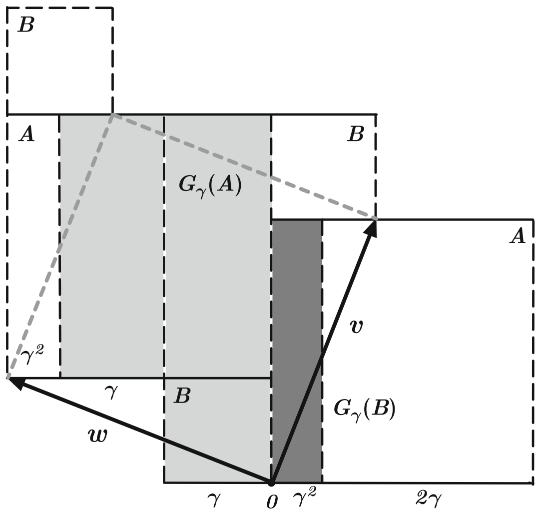

Fig. 1. The generating Markov partition $\mathcal{M}_{G_\gamma}$ and the dynamics of the Anosov automorphism G_γ for the case $a = 2$. The point $0 = \pi_\gamma(0, 0)$ is a fixed point of G_γ

the rectangles of the generating Markov partition $\mathcal{M}_{G_\gamma}$ of G_γ (see Sect. 1). Every G topologically conjugateto G_γ determines exactly a marked pairs $(G, z_1), \ldots, (G, z_a)$, where $z_1, \ldots, z_a$ are the fixed points of G.

We say that two marked C^{1+} Anosov diffeomorphisms $(G_0, z_0), (G_1, z_1) \in \mathcal{G}_\gamma$ are *marked smooth conjugate* if the topological conjugacy h between G_0 and G_1, with $h(z_0) = z_1$, is a C^{1+} diffeomorphism. The *marked smooth conjugacy class* of $(G_0, z_0) \in \mathcal{G}_\gamma$ consists of all $(G_1, z_1) \in \mathcal{G}_\gamma$ that are marked smooth conjugate to (G_0, z_0). Hence, two C^{1+} Anosov diffeomorphisms G_0 and G_1 are smooth conjugate, if there are fixed points z_0 of G_0 and z_1 of G_1 such that (G_0, z_0) and (G_1, z_1) are marked smooth conjugate.

2.2. The rigid rotation g. Let $\pi_\gamma : \mathbb{R}^2 \to \mathbb{T}_\gamma$ be the natural projection, where $\mathbb{T}_\gamma = \mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$. Let $\mathbb{S} = \mathbb{R}/[1+\gamma]\mathbb{Z}$ be the anticlockwise oriented circle with the metric induced by the Euclidean metric on $\mathbb{R}$. Let $\pi_\mathbb{S} : \mathbb{R} \to \mathbb{S}$ be the natural projection with the property that

$$\pi_\mathbb{S}(x) = \pi_\mathbb{S}(x + 1 + \gamma),$$

for every $x \in \mathbb{R}$. Let $i_\mathbb{S} : \pi_\gamma([-\gamma, 1] \times \{0\}) \to \mathbb{S}$ be the natural local inclusion with the property that

$$i_\mathbb{S} \circ \pi_\gamma(x, 0) = \pi_\mathbb{S}(x).$$

We observe that $\pi_\gamma(-\gamma, 0) \neq \pi_\gamma(1, 0)$, but

$$i_\mathbb{S} \circ \pi_\gamma(-\gamma, 0) = \pi_\mathbb{S}(-\gamma) = \pi_\mathbb{S}(1) = i_\mathbb{S} \circ \pi_\gamma(1, 0).$$

Let $g : \mathbb{S} \to \mathbb{S}$ be the rigid rotation with anticlockwise rotation number $\gamma/(1+\gamma)$. The map g has the property that

$$g \circ \pi_\mathbb{S}(x) = \pi_\mathbb{S}(x - \gamma),$$

for every $x \in \mathbb{R}$.

3. Combinatorics and Geometry of the Tilings

In this section, we introduce the tilings and we show the relation between the combinatorics of the tilings and the geometry connected with the Markov partitions of the Anosov diffeomorphims.

3.1. γ-Fibonacci decomposition. The *γ-Fibonacci sequence* is the sequence of natural numbers $(F_n)_{n\in\mathbb{N}}$ defined recursively by

$$F_0 = 1, \; F_1 = 1, \quad \text{and} \quad F_{n+2} = aF_{n+1} + F_n, \quad \text{for} \quad n \geq 0.$$

The *γ-Fibonacci decomposition* $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$ of $i \in \mathbb{N}\backslash\{1\}$ is given as follows (see the greedy algorithm in [8]): (i) F_{n_0} and $a_{n_0} \in \{1, \ldots a\}$ are respectively the largest term in the γ-Fibonacci sequence and the largest integer such that the following inequality holds

$$a_{n_0} F_{n_0} \leq i;$$

(ii) proceeding inductively, for $k \in \mathbb{N}$, let

$$F_{n_0} > \cdots > F_{n_{k-1}}$$

and $a_0, \ldots, a_{n_{k-1}} \in \{1, \ldots, a\}$ be such that

$$a_{n_0} F_{n_0} + \cdots + a_{n_{k-1}} F_{n_{k-1}} \leq i;$$

(iii) choose the largest term in the γ-Fibonacci sequence

$$1 + a = F_2 \leq F_{n_k} < F_{n_{k-1}}$$

and the largest integer $a_{n_k} \in \{1, \ldots, a\}$ such that the following inequality holds

$$a_{n_k} F_{n_k} \leq i - (a_{n_0} F_{n_0} + \cdots + a_{n_{k-1}} F_{n_{k-1}});$$

(iv) hence, there exists a natural number $q \in \mathbb{N}$, such that

$$i = a_{n_0} F_{n_0} + \cdots + a_{n_q} F_{n_q} + b,$$

where $n_q \geq 2$ and $b \in \{0, 1, \ldots a\}$; (v) now, we define the remaining terms of the γ-Fibonacci decomposition of i as follows:

(a) if $b = 0$ then $p = q$;
(b) if $b = 1$ and n_q is odd then $p = q + 1$, $n_p = 0$ and $a_{n_p} = 1$;
(c) if $b = 1$ and n_q is even then $p = q + 1$, $n_p = 1$ and $a_{n_p} = 1$; and
(d) if $b > 1$ then $p = q + 2$, $n_{p-1} = 1$, $a_{n_{p-1}} = b - 1$, $n_p = 0$ and $a_{n_p} = 1$.

Formally, we attribute to 0 the γ-Fibonacci decomposition $(1, F_0)$ and to 1 the γ-Fibonacci decomposition $(1, F_1)$. We observe that F_0 has a bivalent interpretation in this paper, F_0 denotes 0, but values 1 when used in mathematical operations. Hence, the sequence $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$ is a γ-Fibonacci decomposition corresponding to some natural number $i \in \mathbb{N}_0$ if, and only if, for every $j \in \{0, \ldots, p\}$,

(i) $a_{n_j} \in \{1, \ldots, a\}$;
(ii) if $a_{n_j} = a$, then $n_{j+1} \geq n_j + 2$;
(iii) if $n_p = 0$, then $a_{n_p} = 1$;
(iv) if $n_p = 0$ and $n_{p-1} > 1$, then n_{p-1} is odd; and
(v) if $n_p = 1$, then $a_{n_p} = 1$ and n_{p-1} is even.

We observe that every natural number $i \in \mathbb{N}_0$ has a unique γ-Fibonacci decomposition. The *γ-Fibonacci shift* $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ is defined by

$$\sigma(i) = a_{n_0} F_{n_0+1} + \cdots + a_{n_p} F_{n_p+1}.$$

Hence, if $i \in \mathbb{N}$ has γ-Fibonacci decomposition $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$, then $\sigma(i)$ has γ-Fibonacci decomposition $((a_{n_0}, F_{n_0+1}), \ldots, (a_{n_p}, F_{n_p+1}))$.

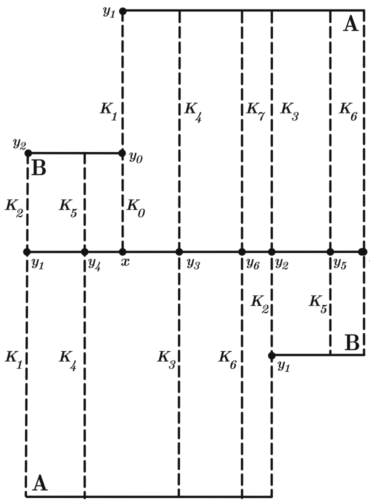

Fig. 2. The unstable leaf $W = \cup_{i \geq 0} K_i$ and the boundary points y_{i-1} and y_i of the unstable spanning leafs K_i, for $i \geq 2$. We note that K_0 is the *right unstable boundary* of the generating Markov *rectangle* B, with endpoints x and y_0; K_1 is the *left unstable boundary* of the generating Markov *rectangle* A, with endpoints x and y_1, where $x = \pi_\gamma(0, 0)$ is the marked fixed point of G_γ

3.2. Combinatorics and geometry of the tilings. Let $W_0 = \{(0, y) : y \geq 0\}$ be the positive vertical axis of $\mathbb{R}^2$. Hence, $W = \pi_\gamma(W_0)$ is part of the unstable leaf of the automorphism G_γ (see notations in Sect. 2). The unstable leaf W has only one endpoint $x = \pi_\gamma(0, 0)$ that is a fixed point of G_γ. Let $y_0 = \pi_\gamma(0, \gamma)$, $y_1 = \pi_\gamma(0, 1)$ and $y_2 = \pi_\gamma(0, 1 + \gamma)$. Let $K_0 \subset W$ be the unstable spanning leaf segment with boundary points x and y_0; let $K_1 \subset W$ be the unstable spanning leaf segment with boundary points x and y_1; and let $K_2 \subset W$ be the unstable spanning leaf segment with boundary points y_1 and y_2. Hence, $K_0 \subset K_1$; K_0 is the right unstable boundary of the generating Markov rectangle B; K_1 is the left unstable boundary of the generating Markov rectangle A; K_2 is the left unstable boundary of the generating Markov rectangle B; and $(K_1 \cup K_2) \setminus K_0$ is the right unstable boundary of the generating Markov rectangle A. Let $y_3,\ y_4, \ldots \in \mathbb{T}_\gamma$ and $K_3,\ K_4, \ldots \in W$ be defined, inductively, as follows: for every $i \geq 3$, (i) K_i is an unstable spanning leaf of a generating Markov rectangle; and (ii) $\{y_{i-1}\} = K_{i-1} \cap K_i$ is a common boundary point of both K_{i-1} and K_i (see Fig. 2). Hence, $W = \cup_{i \geq 1} K_i$, $i_{\mathbb{S}}(y_0) = i_{\mathbb{S}}(y_1)$, $G_\gamma(y_0) = y_1$, but $G_\gamma(y_1) \neq y_2$ because $a > 1$.

The *tiling* $\mathbb{L}_\gamma$ is the set

$$\mathbb{L}_\gamma = \{(K_i, K_{i+1}),\ i \in \mathbb{N}\}.$$

Definition 1. In the tiling $\mathbb{L}_\gamma$, for every $i \in \mathbb{N}$,

(i) if $K_i \subset A$ and $K_{i+1} \subset B$, then i is of type (A, B);
(ii) if $K_i \subset B$ and $K_{i+1} \subset A$, then i is of type (B, A); and
(iii) if $K_i, K_{i+1} \subset A$, then i is of type (A, A).

For simplicity of notation, we denote $\pi_{\mathbb{S}}(0)$ by 0. Let $z_i = i_{\mathbb{S}}(y_i)$, for every $i \in \mathbb{N}$. Hence, by construction, $z_0 = z_1 = g(0)$ and $z_{i+1} = g(z_i) = g^{i+1}(0)$, for every $i \in \mathbb{N}$. Furthermore,

$$z_i = g^{a_{n_0} F_{n_0} + \ldots + a_{n_p} F_{n_p}}(0)$$

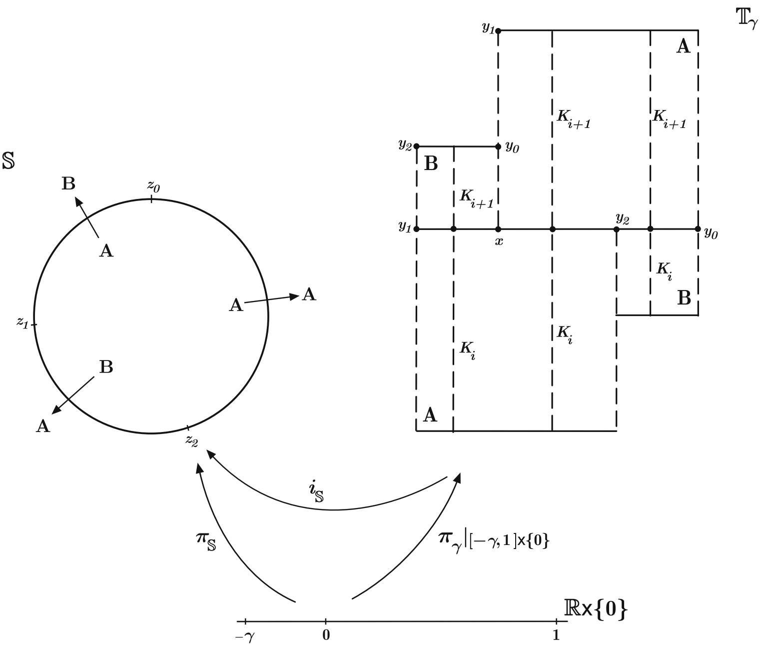

Fig. 3. The map $\pi_{\mathbb{S}}(w) = i_{\mathbb{S}} \circ \pi_\gamma(w, 0)|[-\gamma, 1]$, for every $w \in [-\gamma, 1]$, and the points $z_i = i_{\mathbb{S}}(y_i) = \pi_{\mathbb{S}}(w_i)$, for $i \in \mathbb{N}$. We note that $z_0 = i_{\mathbb{S}}(x) = \pi_{\mathbb{S}}(0)$

where $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$ is the γ-Fibonacci decomposition of $i \in \mathbb{N}_0$ (see Fig. 3). Let us denote by $\ell(x, y_i)$ the leaf segment with endpoints x and y_i. Let $m_A(i)$ be the number of spanning leaf segments of the Markov rectangle A in $\mathbb{L}_\gamma$ that belong to $\ell(x, y_i)$; and let $m_B(i)$ be the number of spanning leaf segments of the Markov rectangle B in $\mathbb{L}_\gamma$ that belong to $\ell(x, y_i)$. Hence,

$$i_{\mathbb{S}}(y_i) = z_i = g^{m_A(i)+m_B(i)}(0).$$

By the hyperbolic properties of G_γ,

$$G_\gamma(\ell(x, y_i)) = \ell(x, G_\gamma(y_i)).$$

Since (i) the image by G_γ of each unstable spanning leaf segment of the Markov rectangle A contains, exactly, one unstable spanning leaf segment of the Markov rectangle B and a unstable spanning leaf segment of the Markov rectangle A and (ii) the image by G_γ of each unstable spanning leaf segment of the Markov rectangle B contains, exactly, one unstable spanning leaf segment of the Markov rectangle A, we conclude the following: $\ell(x, G_\gamma(y_i))$ contains (a) $am_A(i) + m_B(i)$ spanning leaf segments of the Markov rectangle A in $\mathbb{L}_\gamma$ and (b) $m_A(i)$ spanning leaf segments of the Markov rectangle B in $\mathbb{L}_\gamma$. Hence,

$$G_\gamma(y_i) = y_{(a+1)m_A(i)+m_B(i)}.$$

By the hyperbolic properties of G_γ,

$$y_{F_0} = y_0 = \pi_\gamma(1, 0) = \pi_\gamma(0, \gamma).$$

Furthermore, for every $n \in \mathbb{N}$,

$$G_\gamma^n(y_{F_0}) = \pi_\gamma((-\gamma)^n, 0) = \pi_\gamma(0, \gamma^{1-n}).$$

Lemma 1 (Fibonacci embedding). *For every $n \in \mathbb{N}$, the leaf $\ell(x, y_{F_{n+1}})$ contains*

$$m_A(F_{n+1}) = \sum_{k=0}^{n} (-1)^k F_{n+1-k}$$

spanning leaf segments of the Markov rectangle A in $\mathbb{L}_\gamma$;

$$m_B(F_{n+1}) = \sum_{k=0}^{n-1} (-1)^k F_{n-k}$$

spanning leaf segments of the Markov rectangle B in $\mathbb{L}_\gamma$. Furthemore,

$$y_{F_{n+1}} = G_\gamma(y_{F_n}) = G_\gamma^{n+1}(y_{F_0}).$$

For every $n \in \mathbb{N}$, recall that

$$aF_{n+1} = F_{n+2} - F_n.$$

Proof. The leaf $\ell(x, y_{F_2})$ has the property that $m_A(F_2) = a = F_2 - F_1$ and $m_B(F_2) = 1 = F_1$. By induction, let us assume that lemma holds for n and let us prove it for $n+1$. Let m_A denote the number of spanning leaf segments of the Markov rectangle A in $\mathbb{L}_\gamma$ that belong to $\ell(x, G(y_{F_{n+1}}))$. Hence,

$$\begin{aligned}
m_A &= am_A(F_{n+1}) + m_B(F_{n+1}) \\
&= \sum_{k=0}^{n} (-1)^k aF_{n+1-k} + \sum_{k=0}^{n-1} (-1)^k F_{n-k} \\
&= \sum_{k=0}^{n} (-1)^k (F_{n+2-k} - F_{n-k}) + \sum_{k=0}^{n-1} (-1)^k F_{n-k} \\
&= \sum_{k=0}^{n+1} (-1)^k F_{n+2-k}.
\end{aligned}$$

Let m_B denote the number of spanning leaf segments of the Markov rectangle B in $\mathbb{L}_\gamma$ that belong to $\ell(x, G(y_{F_{n+1}}))$. Hence,

$$m_B = m_A(F_{n+1}) = \sum_{k=0}^{n} (-1)^k F_{n+1-k}.$$

Thus, $m_A + m_B = F_{n+2}$ and, so, $G(y_{F_{n+1}}) = y_{F_{n+2}}$. □

The map $w : \mathbb{N}_0 \to [-\gamma, 1]$ is defined implicitly by

$$y_i = \pi_\gamma(w_i, 0).$$

Hence, $z_i = i_{\mathbb{S}}(y_i) = \pi_{\mathbb{S}}(w_i)$ (see Fig. 3) and, so, since the rotation number of g is irrational, the map w is injective and the closure of $w(\mathbb{N}_0)$ is $[-\gamma, 1]$.

Theorem 1 (Fibonacci linking minimal and expanding dynamics). *The map* $w : \mathbb{N}_0 \to [-\gamma, 1]$ *is given by*

$$w_i = \sum_{i=0}^{p} a_{n_i} (-\gamma)^{n_i},$$

where $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$ *is the* γ*-Fibonacci decomposition of* i. *Furthermore, for every* $i \in \mathbb{N}_0$,

$$G(y_i) = y_{\sigma(i)}.$$

Proof. For every $i \in \mathbb{N}$ with γ-Fibonacci decomposition $((a_0, F_{n_0}), \ldots, (a_p, F_{n_p}))$, we have

$$z_i = g^i(0) = g^{a_0 F_{n_0} + \cdots + a_p F_{n_p}}(0). \tag{1}$$

By Lemma 1, $g^{F_i}(0) = z_{F_i} = \pi_{\mathbb{S}}((-\gamma)^i)$. Since g is the rigid rotation, for every $x \in \mathbb{R}$ and $i \in \mathbb{N}$, we have

$$g^{F_i}(\pi_{\mathbb{S}}(x)) = \pi_{\mathbb{S}}(x + (-\gamma)^i). \tag{2}$$

Putting together equalities (1) and (2), we obtain that

$$g^{a_0 F_{n_0} + \cdots + a_p F_{n_p}}(0) = \pi_{\mathbb{S}}\left(\sum_{i=0}^{p} a_{n_i} (-\gamma)^{n_i}\right).$$

Therefore, $w_i = \sum_{j=0}^{p} a_{n_j} (-\gamma)^{n_j} \in [-\gamma, 1]$. Since

$$i_{\mathbb{S}}(G(y_i)) = \pi_{\mathbb{S}}(-\gamma w_i) = \pi_{\mathbb{S}}(w_{\sigma(i)}) = i_{\mathbb{S}}(y_{\sigma(i)}),$$

we obtain that $G(y_i) = y_{\sigma(i)}$. □

In the next theorem, we use the γ-Fibonacci decomposition introduced in Sect. 3.

Theorem 2 (Combinatorial-geometric algorithm). *For every* $i \in \mathbb{N}$ *with* γ*-Fibonacci decomposition* $((a_{n_0}, F_{n_0}), \ldots, (a_{n_p}, F_{n_p}))$,

(i) *i is of type (A, B), if n_p is odd; and*
(ii) *i is of type (A, A), if ($n_p \geq 2$ and n_p is even) or ($n_p = 0$, $n_{p-1} = 1$, $1 < a_{n_{p-1}} < a$) or ($n_p = 0$, $n_{p-1} = 1$, $a_{n_{p-1}} = 1$ and n_{p-2} is odd).*
(iii) *i is of type (B, A), if either ($n_p = 0$ and $n_{p-1} > 1$) or ($n_p = 0$, $n_{p-1} = 1$, $a_{n_{p-1}} = 1$ and n_{p-2} is even);*

Hence, the type of $i \in \mathbb{N}$ is fully determined by its γ-Fibonacci decomposition.

Proof. Recall that $a\gamma = 1 - \gamma^2$, and so $a\gamma \sum_{i \geq 0} \gamma^{2i} = 1$.

There are seven distinct cases to consider depending upon the γ-Fibonacci decomposition $((a_0, F_{n_0}), \ldots, (a_p, F_{n_p}))$ of $i \in \mathbb{N}$ (see Fig. 4). In each case, we will determine w_L and w_R such that $w_L \leq w_i \leq w_R$.

(i) $n_p = 1$ and, so, $n_{p-1} \geq 2$ is even. Hence,

$$w_L = -\gamma + \gamma^{n_{p-1}} - (a-1)\gamma^{n_{p-1}+1} - a \sum_{i \geq 0} \gamma^{n_{p-1}+3+2i} = -\gamma + \gamma^{n_{p-1}+1} \geq -\gamma$$

and

$$w_R = -\gamma + (a-1)\gamma^2 + a \sum_{i \geq 1} \gamma^{2+2i} = -\gamma^2.$$

Thus, w_i is of type (A, B).

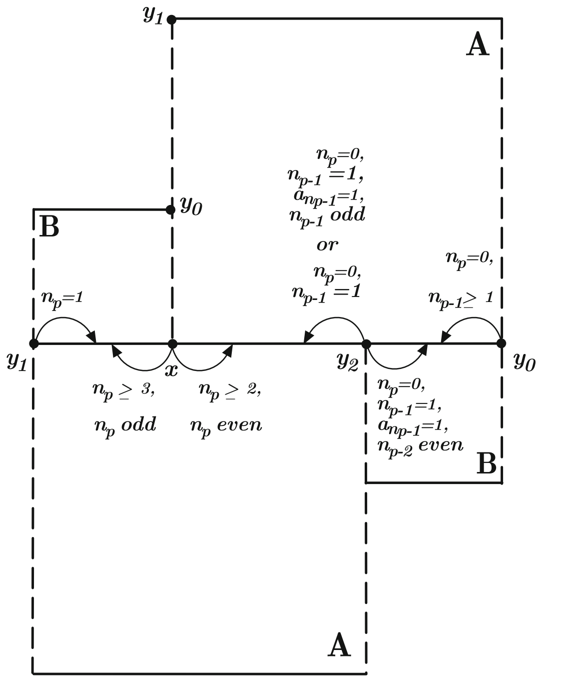

Fig. 4. The location of the point y_i depending upon the γ-Fibonacci decomposition of $i \in \mathbb{N}$. Here we consider the case $a = 2$

(ii) $n_p \geq 3$ and n_p is odd. Hence,

$$w_L = -a \sum_{i \geq 0} \gamma^{3+2i} = -\gamma^2.$$

and

$$w_R = -\gamma^{n_p} + (a-1)\gamma^{n_p+1} + a \sum_{i \geq 0} \gamma^{n_p+3+2i} = -\gamma^{n_p+1} < 0$$

Thus, w_i is of type (A, B).

(iii) $n_p \geq 2$ and n_p is even. Hence,

$$w_L = \gamma^{n_p} - (a-1)\gamma^{n_p+1} - a \sum_{i \geq 0} \gamma^{n_p+3+2i} = \gamma^{n_p+1} > 0$$

and

$$w_R = a \sum_{i \geq 0} \gamma^{2+2i} = \gamma.$$

Thus, w_i is of type (A, A).

(iv) $n_p = 0, n_{p-1} = 1, 1 < a_{n_{p-1}} < a$. Hence,

$$w_L = 1 - (a-1)\gamma - a \sum_{i \geq 0} \gamma^{3+2i} = \gamma$$

and

$$w_R = 1 - 2\gamma + (a-1)\gamma^2 + a \sum_{i \geq 1} \gamma^{2+2i} = 1 - \gamma - \gamma^2.$$

Thus, w_i is of type (A, A).

(v) $n_p = 0, n_{p-1} = 1, a_{n_{p-1}} = 1$ and n_{p-2} is odd. Hence,

$$w_L = 1 - \gamma - a \sum_{i \geq 0} \gamma^{3+2i} = 1 - \gamma - \gamma^2$$

and

$$\begin{aligned} w_R &= 1 - \gamma - \gamma^{n_{p-2}} + (a-1)\gamma^{n_{p-2}+1} \\ &\quad + a \sum_{i \geq 0} \gamma^{n_{p-2}+3+2i} = 1 - \gamma - \gamma^{n_{p-2}+1} \leq 1 - \gamma. \end{aligned}$$

Thus, w_i is of type (A, A).

(vi) $n_p = 0, n_{p-1} = 1, a_{n_{p-1}} = 1$ and n_{p-2} is even. Hence,

$$\begin{aligned} w_L &= 1 - \gamma + \gamma^{n_{p-2}} - (a-1)\gamma^{n_{p-2}+1} - a \sum_{i \geq 0} \gamma^{n_{p-2}+3+2i} \\ &= 1 - \gamma + \gamma^{n_{p-2}+1} \geq 1 - \gamma \end{aligned}$$

and

$$w_R = 1 - \gamma + (a-1)\gamma^2 + a \sum_{i \geq 1} \gamma^{2+2i} = 1 - \gamma^2.$$

Thus, w_i is of type (B, A).

(vii) $n_p = 0$ and $n_{p-1} \geq 3$, and so n_{p-1} is odd. Hence,

$$w_L = 1 - a \sum_{i \geq 0} \gamma^{3+2i} = 1 - \gamma^2$$

and

$$w_R = 1 - \gamma^{n_{p-1}} + (a-1)\gamma^{n_{p-1}+1} + a \sum_{i \geq 0} \gamma^{n_{p-1}+3+2i} = 1 - \gamma^{n_{p-1}+1} \leq 1.$$

Thus, w_i is of type (B, A). □

4. γ-Tilings

The γ-tilings are affine tilings satisfying the exponentially fast γ-Fibonacci repetitive condition, the matching condition and the boundary condition that we introduce in this section. In particular, we prove that the affine tilings determined by a marked Anosov diffeomorphims (G, z) are γ-tilings.

Let $|I|$ be the length of the unstable leaf segment I with respect to a Riemannian metric on $\mathbb{T}$. An *affine* tiling consists of a positive sequence $(a_i)_{i\in N}$ that determines the length of the ratios of the tiling $\mathbb{L}_\gamma$

$$a_i = \frac{|K_{i+1}|}{|K_i|},$$

i.e. an affine structure for the tiling $\mathbb{L}_\gamma$ (see Sect. 3). Let $\mathcal{A}_\gamma$ be the set of all positive sequences $(a_i)_{i\in N}$, i.e. the set of all affine structures for the tiling $\mathbb{L}_\gamma$.

Lemma 2 (Realized affine tilings). *The map $\underline{\tau} : \mathcal{G}_\gamma \to \mathcal{A}_\gamma$ that associates to each pair $(G, z) \in \mathcal{G}_\gamma$ the positive sequence $\underline{\tau}(G, z) = (\tau_i(G, z))_{i\in\mathbb{N}}$ given by*

$$\tau_i(G, z) = \lim_{n\to\infty} \frac{|G^{-n}(h_G(K_{i+1}))|}{|G^{-n}(h_G(K_i))|} \tag{3}$$

is well-defined.

The proof of Lemma 2 follows from the control ratio distortion lemma (see Lemma 3.6 in [26]).

Remark 1. By the above lemma, the Anosov automorphism G_γ determines the following affine *rigid* tiling $\underline{\tau}(G_\gamma, \pi_\gamma(0, 0)) = (\tau_{\gamma,i})_{i\in\mathbb{N}}$:

(i) $\tau_{\gamma,i} = \gamma$, if i is of type (A, B);
(ii) $\tau_{\gamma,i} = \gamma^{-1}$, if i is of type (B, A);
(iii) $\tau_{\gamma,i} = 1$, if i is of type (A, A).

4.1. Matching condition. A sequence $(\tau_i)_{i\in\mathbb{N}}$ satisfies the *matching condition* if, for every $i \in \mathbb{N}$, the following conditions hold (see Fig. 5):

(i) if i is of type (B, A), then

$$\tau_i = \tau_{\sigma(i)} \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \tau_{\sigma(i)+k}\right).$$

(ii) if i is of type (A, B), then

$$\tau_i = \tau_{\sigma(i)} \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \tau^{-1}_{\sigma(i)-k}\right)^{-1}.$$

(iii) if i is of type (A, A), then

$$\tau_i = \tau_{\sigma(i)} \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \tau^{-1}_{\sigma(i)-k}\right)^{-1} \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \tau_{\sigma(i)+k}\right).$$

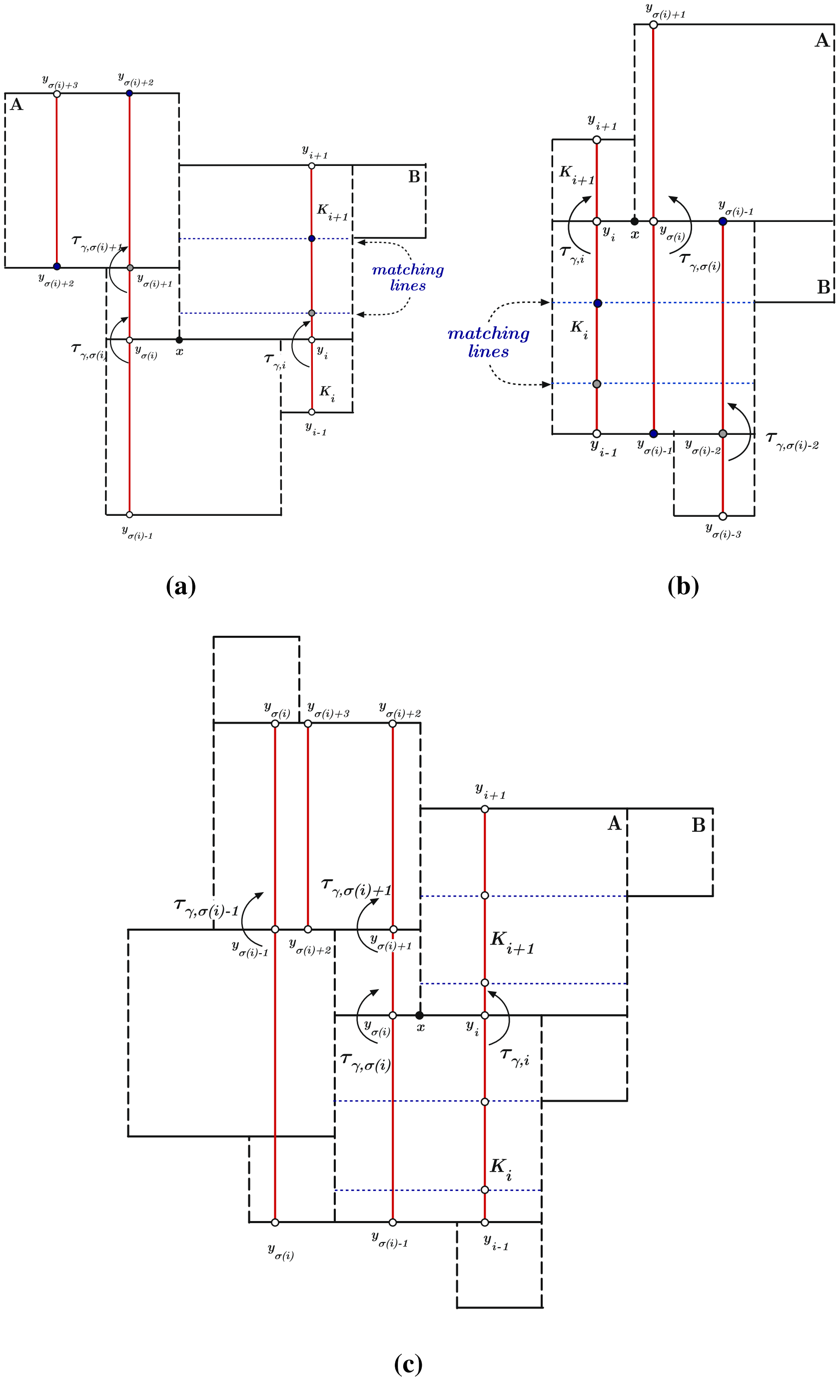

Fig. 5. The matching condition for the γ-sequence $(\tau_{\gamma,i})_{i\in\mathbb{N}} = \tau(G_\gamma, \pi_\gamma(0,0))$ with the three possible cases for $i \in \mathbb{N}$. Here we consider the case $a = 2$. **a** Case i of type (B, A). **b** Case i of type (A, B). **c** Case i of type (A, A)

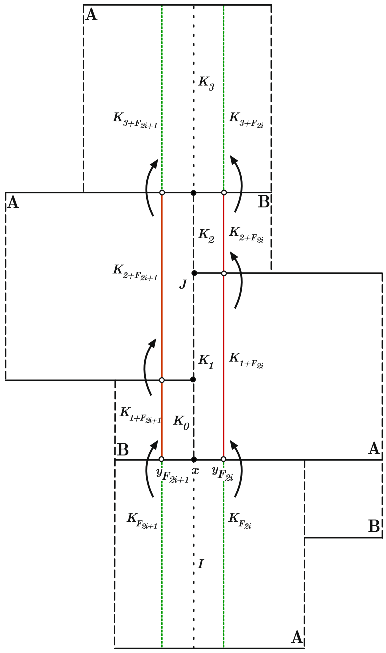

Fig. 6. The boundary condition for the γ-sequence $(\tau_{\gamma,i})_{i\in\mathbb{N}}$. Here, $J = (K_1 \cup K_2)\backslash K_0$ is the right boundary of the Markov rectangle A and I is the unstable spanning leaf segment in the interior of A such that $I \cap K_0 = \{x\}$

Lemma 3. *For every* $(G, z) \in \mathcal{G}_\gamma$, *the sequence* $\underline{\tau}(G, z)$ *satisfies the matching condition.*

Proof. Putting together Lemma 2 and Eq. (4) in Appendix 6, we obtain that $\tau_{G,i} = \sigma_G(K_i : K_{i+1})$, for every $i \in \mathbb{N}$. Hence Lemma 3 follows from Lemma 5 in Appendix 6. $\square$

We note that every sequence $(\beta_i)_{i\in\mathbb{N}\backslash\sigma(\mathbb{N})}$ determines, uniquely, a sequence $(\tau_i)_{i\in\mathbb{N}}$ satisfying the matching condition as follows: (i) for every $i \in \mathbb{N}\backslash\sigma(\mathbb{N})$, define $\tau_i = \beta_i$ and (ii) for every $i \in \sigma(\mathbb{N})$, inductively in i, define $\tau_{\sigma(i)}$ using the matching condition, i.e. τ_i, that was previously defined inductively, and elements $\tau_k = \beta_k \in \mathbb{N}\backslash\sigma(\mathbb{N})$.

4.2. Boundary condition. A sequence $(\tau_i)_{i\in\mathbb{N}}$ satisfies the *boundary condition*, if the following limits are well-defined and satisfy the conditions:

$$\text{(i)} \quad \lim_{i\to+\infty} \tau_{F_i+2}^{-1}\left(1 + \tau_{F_i+1}^{-1}\right) \neq 0$$
$$\text{(ii)} \quad \lim_{i\to+\infty} \tau_{F_i}\left(1 + \tau_{F_i+1}\right) \neq 0$$

Following the proof of Lemma 3.6 in [21] we get that, for every $(G, z) \in \mathcal{G}_\gamma$, the sequence $\underline{\tau}(G, z)$ satisfies the boundary condition (see Fig. 6).

4.3. Exponentially fast γ-Fibonacci repetitive condition. We say that a sequence $(\tau_i)_{i\in\mathbb{N}}$ satisfies the *exponentially fast γ-Fibonacci repetitive condition*, if there exist constants

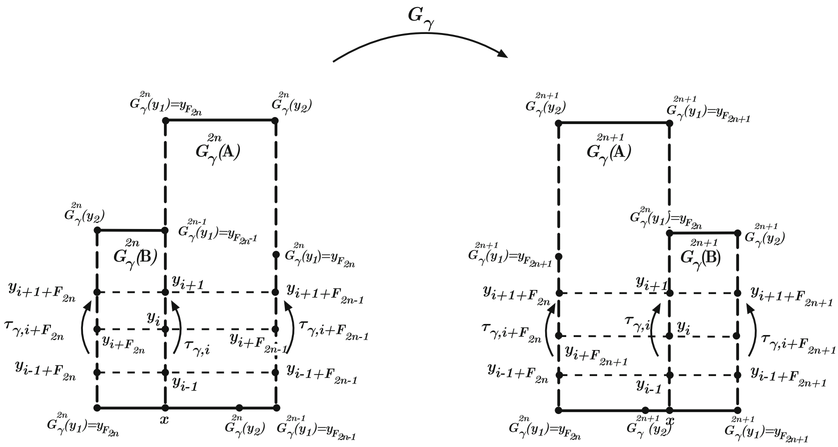

Fig. 7. The sequence $(\tau_{G,i})_{i\in\mathbb{N}}$ satisfies the exponentially fast γ-Fibonacci repetitive property. We observe that $\lim_{n\to\infty} y_{F_n} = y_0$

$C \geq 0$ and $0 < \mu < 1$ such that

$$\left|\tau_{i+F_m} - \tau_i\right| \leq C\mu^m,$$

for every $m \geq 5$ and $3 \leq i < F_{m-1}$ and, also, for $i \in \{1, 2\}$ if m is even.

Following the proof of Lemma 3.7 in [21] we get that, for every $(G, z) \in \mathcal{G}_\gamma$, the sequence $\underline{\tau}(G, z)$ satisfies the exponentially fast γ-Fibonacci repetitive condition (see Fig. 7).

4.4. γ-Tilings. A *γ-tiling* is a sequence $(\tau_i)_{i\in\mathbb{N}}$ of positive numbers satisfying the exponentially fast γ-Fibonacci repetitive condition, the matching condition and the boundary condition. Let $\mathcal{T}_\gamma$ be the set of all γ-tilings.

Since, for every $(G, z) \in \mathcal{G}_\gamma$, the sequence $\underline{\tau}(G, z)$ satisfies the exponentially fast γ-Fibonacci repetitive condition, the boundary condition and the matching condition, we obtain that $\underline{\tau}(G, z)$ is a γ-tiling.

Theorem 3. *The map $\underline{\tau} : \mathcal{G}_\gamma \to \mathcal{T}_\gamma$ determines a one-to-one correspondence between marked smooth conjugacy classes of Anosov diffeomorphisms in $\mathcal{G}_\gamma$ and γ-tilings.*

The proof of Theorem 3 follows as the proof of Theorem 1.1 in [21].

Theorem 3 implies the existence of an infinite dimensional space of well-characterized γ-sequences. However, we are only able to explicit the rigid γ-sequence.

5. Tilings Rigidity

We will give conditions in the regularity of the holonomies of the marked Anosov diffeomorphims such that the corresponding affine tilings are rigid.

Let $G \in \mathcal{G}$ be a C^{1+} Anosov diffeomorphism topologically conjugate to G_γ by a homeomorphism h. Recall that $\mathcal{M}_G = h(M_\gamma) = \{h(A), h(B)\}$ is a Markov partition of G. Let $M, N \in \mathcal{M}_G$ be two Markov rectangles, and let the points $x \in int(M)$ and $y \in int(N)$. We say that x and y are *unstable holonomically related* if there is a stable leaf segment $\ell^u(x, y)$ such that (i) x and y are the endpoints of $\ell^s(x, y)$, i.e. $\partial\ell^s(x, y) = \{x, y\}$, and (ii) $\ell^s(x, y) \subset \ell^s(x, M) \cup \ell^s(y, N)$. Let $P = P_{\mathcal{M}}$ be the set of all pairs (M, N) with unstable holonomically related points. For every Markov rectangle $M \in \mathcal{M}_G$, choose a point $x \in int(M)$ and consider the unstable spanning leaf segment $\ell^u(x, M)$. Let $\mathcal{I} = \{\ell^u_M : M \in \mathcal{M}\}$. For every pair $(M, N) \in P$ there exist maximal unstable leaf segments $\ell^D_{(M,N)} \subset \ell^u_M$ and $\ell^C_{(M,N)} \subset \ell^u_N$ such that the unstable holonomy $\theta_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ is well-defined. We call such holonomies $\theta_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ the *unstable primitive holonomies* associated to the Markov partition $\mathcal{M}_G$. The *complete set of unstable holonomies* $\mathcal{H}_G$ consists of all unstable primitive holonomies and their inverses (see [26,28]).

A diffeomorphism $\theta : I \to J$ is said to be $C^{1+zygmund}$, where I and J are unstable leaf segments, if θ is C^1 and the derivative θ' satisfies the zygmund condition, i.e. for all points $x, y \in I$,

$$\left|\theta'(x) + \theta'(y) - 2\theta'\left(\frac{x+y}{2}\right)\right| = \chi_\theta\left(|y - x|\right),$$

where the function χ_θ is such that $\chi_\theta(t) \to 0$ when $t \to 0$. In particular, a $C^{2+\beta}$ diffeomorphism, with $\beta > 0$, is a $C^{1+zygmund}$ diffeomorphism. The importance of this smooth class follows from the fact that it corresponds to maps that distort cross-ratios of quadruples of points in I by an amount that is $o(|I|)$ (see [20,30]).

Definition 2. A complete set of unstable holonomies $\mathcal{H}_G$ is $C^{1+zygmund}$, if all holonomies $\theta \in \mathcal{H}_G$ are $C^{1+zygmund}$ with respect to the unstable lamination atlas $\mathcal{L}^u(G, \rho)$.

Theorem 4 (Rigidity). *Every $(G, z) \in \mathcal{G}_\gamma$ with a $C^{1+zygmund}$ complete system of unstable holonomies determines a rigid γ-tiling $\underline{\tau}(G_\gamma, z) = \underline{\tau}_\gamma$.*

The proof of Theorem 4 follows as the proof of Theorem 1.3 in [21].

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6. Appendix: Solenoid Functions

An HR-structure determines a pair of stable and unstable solenoid functions, and vice-versa (see Lemma 3.3 in book [28]). In this paper, we are only interested in studying the unstable ratio functions and so we are only going to characterize the unstable solenoid functions. Since the Anosov diffeomorphisms have an invariant measure absolutely continuous with respect to Lebesgue, by duality the unstable solenoid function will determine the stable solenoid function and vice-versa.

Let **sol** denote the set of all ordered pairs (I, J) of unstable spanning leaf segments of the Markov rectangles A and B of G_γ such that the intersection of I and J consists of a single endpoint. Since the set **sol** is topologically a finite disjoint union of disjoint intervals, i.e. the disjoint union of a stable spanning leaf segment of A and a stable spanning leaf segment of B, it has a natural topological structure (see Pinto and Rand [22]).

Definition 3. A function $\sigma : \mathbf{sol} \to \mathbb{R}^+$ is an *(unstable) solenoid function*, if the following conditions hold: (i) σ is Hölder continuous; (ii) σ satisfies the matching condition; and (iii) σ satisfies the boundary condition.

In the next subsections, we define the properties mentioned in the above definition.
Let G be a C^{1+} Anosov diffeomorphism in $\mathcal{G}$. The *(unstable) realized solenoid function* $\sigma_G : \mathbf{sol} \to \mathbb{R}^+$ is well-defined by

$$\sigma_G(I : J) = \lim_{n\to\infty} \frac{|G^{-n}(h_G(J)|}{|G^{-n}(h_G(I))|} \tag{4}$$

by the control ratio distortion lemma (see [22]).

Theorem 5. *The map $G \to \sigma_G$ determines a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms $G \in \mathcal{G}_\gamma$, that have an invariant measure absolutely continuous with respect to Lebesgue, and solenoid functions.*

Proof. By Theorem 3.4 in [28] there is a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms and pairs of stable and unstable solenoid function. By Theorem 8.17 in [28], an unstable measure ratio function determines a unique dual stable measure function and vice-versa. For Anosov diffeomorphisms, that have an invariant measure absolutely continuous with respect to Lebesgue, the stable and unstable measure ratio functions are dual and by Lemma 8.5 in [28], there is a one-to-one correspondence between stable (resp. unstable) measure ratio functions and stable (resp. unstable) solenoid functions. Since, the stable function is obtained by duality from the unstable function we obtain the result. □

6.1. Hölder continuity of solenoid functions. We define a pseudo-metric $d_{\mathbf{sol}} : \mathbf{sol} \times \mathbf{sol} \to \mathbb{R}^+$ on the set **sol** as follows: (i) if $I = I'$ and $J = J'$ then

$$d_{\mathbf{sol}}\left((I, J), (I', J')\right) = 0;$$

(ii) if $I \cap I' = \emptyset$ and $J \cap J' = \emptyset$ then

$$d_{\mathbf{sol}}\left((I, J), (I', J')\right) = \max\left\{d\left(I, I'\right), d\left(J, J'\right)\right\};$$

(iii) otherwise,

$$d_{\mathbf{sol}}\left((I, J), (I', J')\right) = 1.$$

The solenoid function σ is *Hölder continuity* if, for all $t = (I, J)$ and $t' = (I', J')$ in **sol**, the solenoid function σ satisfies

$$\left|\sigma(t) - \sigma(t')\right| \le \mathcal{O}\left(\left(d_{\mathbf{sol}}\left(t, t'\right)\right)^\alpha\right),$$

for some $\alpha > 0$.

6.2. Boundary condition. Let $(I_i, I_{i+1}), (J_j, J_{j+1}) \in \mathbf{sol}$, for each $i \in \{0, \ldots, m\}$ and each $j \in \{0, \ldots, n\}$ with the following properties: (i) $I_0 = J_0$, (ii) $\cup_{i=1}^m I_i = \cup_{j=1}^n J_j$ and (iii) $I_i \neq J_i$ for some $i \in 1, \ldots, m$. Then the following two ratios are equal

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \frac{|I_j|}{|I_{j-1}|} = \frac{|\cup_{i=1}^m I_i|}{|I_0|} = \frac{|\cup_{j=1}^n J_j|}{|J_0|} = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{|J_j|}{|J_{j-1}|}.$$

We observe that the unstable spanning leaf segments $I_1, \ldots, I_m$ and $J_1, \ldots, J_n$ must be boundaries of Markov rectangles. Thus, a realized solenoid function σ_G must satisfy the following *boundary condition* for all such leaf segments:

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \sigma_G \left(I_{j-1} : I_j\right) = \sum_{i=1}^{n} \prod_{j=1}^{i} \sigma_G \left(J_{j-1} : J_j\right). \tag{5}$$

Let K_0, K_1, K_2 and K_3 be the unstable spanning leaf segments as defined in Sect. 3. Recall that, K_0 is the unstable spanning leaf segment of the right boundary of the Markov rectangle B; $J = (K_1 \cup K_2) \setminus K_0$ is the unstable spanning leaf segment of the right boundary of the Markov rectangle A; K_1 is the unstable spanning leaf segment of the left boundary of the Markov rectangle A; K_2 is the unstable spanning leaf segment of the left boundary of the Markov rectangle B; and so $K_3 \cap K_2 = K_3 \cap J = \{y_2\}$. Let I be the unstable spanning leaf segment in the interior of A such that $I \cap K_0 = \{x\}$ (see Fig. 8).

Lemma 4. *Let $\sigma_G : \mathbf{sol} \to \mathbb{R}^+$ be a realized solenoid function. Then σ_G satisfies the boundary condition if the following two conditions hold:*

$$\sigma_G(I : K_1)(1 + \sigma_G(K_1 : K_2)) = \sigma_G(I : K_0)(1 + \sigma_G(K_0 : J)); \tag{6}$$

and

$$\sigma_G(K_3 : K_2)(1 + \sigma_G(K_2 : K_1)) = \sigma_G(K_3 : J)(1 + \sigma_G(J : K_0)). \tag{7}$$

Proof. The boundary condition (5) corresponds to

$$\frac{|K_1|}{|I|}\left(1 + \frac{|K_2|}{|K_1|}\right) = \frac{|K_1 \cup K_2|}{|I|} = \frac{|K_0 \cup J|}{|I|} = \frac{|K_0|}{|I|}\left(1 + \frac{|J|}{|K_0|}\right),$$

and

$$\frac{|K_2|}{|K_3|}\left(1 + \frac{|K_1|}{|K_2|}\right) = \frac{|K_1 \cup K_2|}{|K_3|} = \frac{|K_0 \cup J|}{|K_3|} = \frac{|J|}{|K_3|}\left(1 + \frac{|K_0|}{|J|}\right).$$

Hence, the boundary condition for σ_G is given by (6) and (7). $\square$

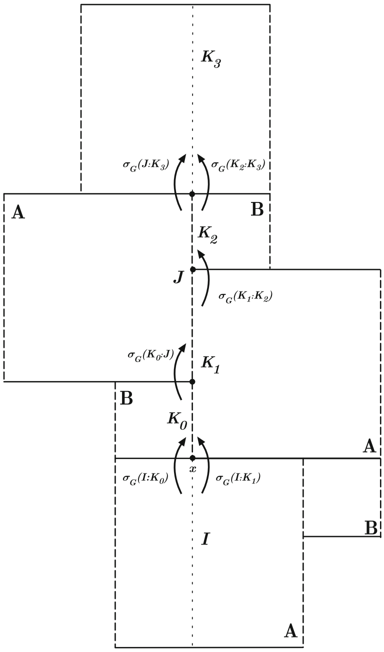

Fig. 8. The boundary condition for the realized solenoid function σ_G

6.3. Matching condition. Let $(I, J) \in \mathbf{sol}$. Suppose that there are pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathbf{sol}$$

such that $G_\gamma I = \bigcup_{j=0}^{k-1} I_j$ and $G_\gamma J = \bigcup_{j=k}^{n-1} I_j$. Then

$$\frac{|G_\gamma J|}{|G_\gamma I|} = \frac{\sum_{j=k}^{n-1} |I_j|}{\sum_{j=0}^{k-1} |I_j|} = \frac{\sum_{j=k}^{n-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}.$$

Hence, the realized solenoid function σ_G must satisfy the *matching condition* for all such leaf segments (see Fig. 5):

$$\sigma_G(I : J) = \frac{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_G(I_{i-1} : I_i)}{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_G(I_{i-1} : I_i)}. \tag{8}$$

Lemma 5. *Let $\sigma_G : \mathbf{sol} \to \mathbb{R}^+$ be a realized solenoid function. For $a \in \mathbb{N}$, the matching condition holds for σ_G if, for every $(K_1, K_2) \in \mathbf{sol}$, the following conditions hold:*

(i) if $K_1, K_2 \in A$, then

$$\sigma_G(K_1 : K_2) = \prod_{k=1}^{a+1} \sigma_G(I_k : I_{k+1}) \left(\frac{1 + \sum_{j=a+2}^{2a+1} \prod_{k=a+2}^{j} \sigma_G(I_k : I_{k+1})}{1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \sigma_G(I_k : I_{k+1})} \right). \tag{9}$$

where $I_1, \ldots, I_{2a+2}$ are such that $G_\gamma(K_1) = \cup_{i=1}^{a+1} I_i$, $G_\gamma(K_2) = \cup_{i=a+2}^{2a+2} I_i$ and $(I_i, I_{i+1}) \in \mathbf{sol}$ for $i \in \{1, \ldots, 2a+1\}$.

(ii) *if* $K_1 \in A$ *and* $K_2 \in B$*, then*

$$\sigma_G(K_1 : K_2) = \prod_{k=1}^{a+1} \sigma_G(I_k : I_{k+1}) \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \sigma_G(I_k : I_{k+1})\right)^{-1} \tag{10}$$

where $I_1, \ldots, I_{a+2}$ *are such that* $G_\gamma(K_1) = \cup_{i=1}^{a+1} I_i$, $G_\gamma(K_2) = I_{a+2}$ *and* $(I_i, I_{i+1}) \in$ **sol** *for* $i \in \{1, \ldots, a+1\}$.

(iii) *if* $K_1 \in B$ *and* $K_2 \in A$*, then*

$$\sigma_G(K_1 : K_2) = \sigma_G(I_1 : I_2) \left(1 + \sum_{j=2}^{a+1} \prod_{k=2}^{j} \sigma_G(I_k : I_{k+1})\right) \tag{11}$$

where $I_1, \ldots, I_{a+2}$ *are such that* $G_\gamma(K_1) = I_1$, $G_\gamma(K_2) = \cup_{i=2}^{a+2} I_i$ *and* $(I_i, I_{i+1}) \in$ **sol** *for* $i \in \{1, \ldots, a+1\}$.

Proof. If $(K_1, K_2) \in$ **sol** then (K_1, K_2) satisfies either condition (i), (ii) or (iii) above (see Fig. 5). Let us check that the formulas (9), (10) and (11) correspond to the matching condition (8) for σ_G.

(i) If $K_1, K_2 \in A$ then there exists $(I_i, I_{i+1}) \in$ **sol**, for $i = 1, \ldots, 2a+2$, such that $G_\gamma(K_1) = I_1 \cup \cdots \cup I_{a+1}$ and $G_\gamma(K_2) = I_{a+2} \cup \cdots \cup I_{2a+2}$. Furthermore,

$$\frac{|G_\gamma(K_2)|}{|G_\gamma(K_1)|} = \frac{\sum_{i=a+2}^{2a+1} |I_i|}{\sum_{i=1}^{a+1} |I_i|} = \prod_{k=1}^{a+1} \frac{|I_{k+1}|}{|I_k|} \left(\frac{1 + \sum_{j=a+2}^{2a+1} \prod_{k=a+2}^{j} \frac{|I_{k+1}|}{|I_k|}}{1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \frac{|I_{k+1}|}{|I_k|}} \right).$$

Hence, equality (9) follows from equality (8).

(ii) If $K_1 \in A$ and $K_2 \in B$ then there exists $(I_i, I_{i+1}) \in$ **sol**, for $i = 1, \ldots, a+1$, such that $G_\gamma(K_1) = I_1 \cup \cdots \cup I_{a+1}$ and $G_\gamma(K_2) = I_{a+2}$. Furthermore,

$$\frac{|G_\gamma(K_2)|}{|G_\gamma(K_1)|} = \frac{|I_{a+2}|}{\sum_{i=1}^{a+1} |I_i|} = \prod_{k=1}^{a+1} \frac{|I_{k+1}|}{|I_k|} \left(1 + \sum_{j=1}^{a} \prod_{k=1}^{j} \frac{|I_{k+1}|}{|I_k|}\right)^{-1}.$$

Hence, equality (10) follows from equality (8).

(iii) If $K_1 \in B$ and $K_2 \in A$, then there exists $(I_i, I_{i+1}) \in$ **sol**, for $i = 1, 2$, such that $G_\gamma(K_1) = I_1$ and $G_\gamma(K_2) = I_2 \cup \cdots \cup I_{a+2}$. Furthermore,

$$\frac{|G_\gamma(K_2)|}{|G_\gamma(K_1)|} = \frac{\sum_{i=2}^{a+2} |I_i|}{|I_1|} = \frac{|I_2|}{|I_1|} \left(1 + \sum_{j=2}^{a+1} \prod_{k=2}^{j} \frac{|I_{k+1}|}{|I_k|}\right).$$

Hence, equality (11) follows from equality (8). □

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