Chapter 33
Complete versus Incomplete Information in the Hotelling Model

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33.1 Introduction

Since the seminal work of Hotelling [14], the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets. The Hotelling model became one of the most important methods of analyzing product differentiation.

In his model, Hotelling present a city represented by a line segment where a uniformly distributed continuum of consumers have to buy a homogeneous good. Consumers have to support linear transportation costs when buying the good in one of the two firms of the city. The firms compete in a two-staged location-price game, where simultaneously choose their location and afterwards set their prices in order to maximize their profits. After this seminal work, a lot of research has been done in related models (see [1–4, 13, 15–21, 24–26]).

In this work, we do not study the Hotelling models in which the location choice by the firms plays a major rule, but models where the location of firms are fixed, this is, models of price competition under spatial nature. We assume that the firms are located at the extremes of the line and so we do not study the first subgame in location strategies. Our main goal is to compare the price formation in the Hotelling

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model with complete and incomplete information in the production costs of both firms. The incomplete information consists in each firm to know its production cost but to be uncertain about the competitor cost as usual in oligopoly theory (see [5–12, 22, 23]).

We introduce the bounded costs BC condition that defines a bound for the production costs in terms only of the exogenous variables that are the transportation cost and the road length of the segment line (see Sect. 33.5). Hence, the prices strategies do not appear in the BC condition. Under the bounded costs BC condition, the Nash price strategy for the firms exists. We explicitly determine for the profit, consumer surplus and welfare, the quantitative economical advantages and disadvantages between having complete or incomplete information in the production costs. We prove that, in expected value, the consumer surplus and the welfare are greater with incomplete information than with the complete information and the difference is determined by the variances of the probability distributions (see Eqs. (33.17) and (33.19)). In expected value, the profit it is greater for the firm with higher variance for the probability distribution of its productions costs with incomplete information than with the complete information. However, in expected value, the profit can be smaller for the firm with lower variance for the probability distribution of its productions costs with incomplete information than with the complete information (see Eqs. (33.14) and (33.15)). We conclude by observing that if the probability distribution of the costs is the same for both firms then, in expected value, the profit it is greater for the firms with incomplete information than with the complete information and the welfare is also greater with incomplete information than with the complete information.

Our results are universal, in the incomplete information scenario, in the sense that they apply to all probability distributions in the production costs. All the results presented in this chapter are proved in [18].

33.2 Hotelling Model

The buyers of a commodity will be supposed uniformly distributed along a line with length l. In the two ends of the line there are two firms A and B, located at positions 0 and l respectively, selling the same commodity with unitary *production costs* c_A and c_B. No customer has any preference for either seller except on the ground of price plus *transportation cost* t. We will assume that each consumer buys a single unit of the commodity, in each unit of time and in each unit of length of the line. Denote A's *price* by p_A and B's *price* by p_B. The point of division $x = x(p_A, p_B) \in]0, l[$ between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or from B.

The point x is the location of the *indifferent consumer* to buy from firm A or firm B, if

$$p_A + tx = p_B + t\,(l - x).$$

Solving for x, we obtain

$$x = \frac{p_B - p_A + t\,l}{2t}.$$

Both firms have a non-empty market share if and only if $x \in]0, l[$. Hence, both firms have a non-empty market share if and only if the prices satisfy

$$|p_A - p_B| < tl. \tag{33.1}$$

Assuming inequality (33.1), both firms A and B have a non-empty demand (x and $l - x$) and the *profits* of the two firms are defined respectively by

$$\pi_A = (p_A - c_A)\,x = (p_A - c_A)\,\frac{p_B - p_A + tl}{2t}; \tag{33.2}$$

and

$$\pi_B = (p_B - c_B)\,(l - x) = (p_B - c_B)\,\frac{p_A - p_B + tl}{2t}. \tag{33.3}$$

Two of the fundamental economic quantities in oligopoly theory are the consumer surplus CS and the welfare W that are computed as follows. Let us denote by v_T be the total amount that consumers are willing to pay for the commodity. The total amount $v(y)$ that a consumer located at y pays for the commodity is given by

$$v(y) = \begin{cases} p_A + ty & \text{if } 0 < y < x; \\ p_B + t(l - y) & \text{if } x < y < l. \end{cases}$$

The *consumer surplus* CS is the difference between the total amount that a consumer is willing to pay v_T and the total amount that the consumer pays $v(y)$

$$CS = \int_0^l v_T - v(y)\mathrm{d}y. \tag{33.4}$$

The *welfare* W is given by adding the profits of firms A and B with the consumer surplus

$$W = CS + \pi_A + \pi_B. \tag{33.5}$$

A price strategy $(\underline{p}_A, \underline{p}_B)$ for both firms is a *Nash* price strategy, if for every deviation of the price $\underline{p}_A$ the profit π_A of firm A decreases, and for every deviation of the price $\underline{p}_B$ the profit π_B of firm B decreases.

Let us compute the Nash price strategy $(\underline{p}_A, \underline{p}_B)$. Differentiating π_A with respect to p_A and π_B with respect to p_B and equalizing to zero, we obtain the first order conditions (FOC). The FOC implies that

$$\underline{p}_A = tl + \frac{1}{3}(2c_A + c_B) \tag{33.6}$$

and

$$\underline{p}_B = tl + \frac{1}{3}(c_A + 2c_B). \tag{33.7}$$

We note that the first order conditions refer to jointly optimizing the profit function (33.2) with respect to the price p_A and the profit function (33.3) with respect to the price p_B.

Since the profit functions (33.2) and (33.3) are concave, the second-order conditions for this maximization problem are satisfied and so the prices (33.6) and (33.7) are indeed maxima for the functions (33.2) and (33.3), respectively. The corresponding equilibrium profits are given by

$$\underline{\pi}_A = \frac{(3tl + c_B - c_A)^2}{18t} \tag{33.8}$$

and

$$\underline{\pi}_B = \frac{(3tl + c_A - c_B)^2}{18t}. \tag{33.9}$$

Furthermore, the consumer indifference location corresponding to the maximizers $\underline{p}_A$ and $\underline{p}_B$ of the profit functions π_A and π_B is

$$\underline{x} = \frac{l}{2} + \frac{c_B - c_A}{6t}.$$

Finally, for the pair of prices $(\underline{p}_A, \underline{p}_B)$ to be a Nash price strategy, we need assumption (33.1) to be satisfied with respect to these pair of prices. We observe that assumption (33.1) is satisfied with respect to the pair of prices $(\underline{p}_A, \underline{p}_B)$ if and only if the following condition with respect to the production costs is satisfied.

Definition 33.1. The Hotelling model satisfies the *bounded costs (BC)* condition, if

$$|c_A - c_B| < 3tl.$$

We note that under the *BC* condition the prices are higher than the production costs $\underline{p}_A > c_A$ and $\underline{p}_B > c_B$. Hence, there is a Nash price strategy if and only if the *BC* condition holds. Furthermore, under the *BC* condition, the pair of prices $(\underline{p}_A, \underline{p}_B)$ is the Nash price strategy.

By Eq. (33.4), the consumer surplus $\underline{CS}$ with respect to the Nash price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\underline{CS} = \int_0^l v_T - v(x)\mathrm{d}x = v_T l - \frac{3}{2}tl^2 - \frac{c_A + 2c_B}{3}l + \frac{(c_B - c_A + 3tl)^2}{36t}. \quad (33.10)$$

By Eq. (33.5), the welfare $\underline{W}$ is given by

$$\underline{W} = v_T l - \frac{1}{4}tl^2 - \frac{c_A + c_B}{2}l + \frac{5(c_A - c_B)^2}{36t}. \quad (33.11)$$

33.3 Uncertainty in the Production Costs

In this section, we introduce a simple notation that is fundamental for the elegance and understanding of the effects of uncertainty in the production costs of both firms.

Let the triples (I_A, Ω_A, q_A) and (I_B, Ω_B, q_B) represent (finite, countable or uncountable) sets of types I_A and I_B with σ-algebras Ω_A and Ω_B and probability measures q_A and q_B, over I_A and I_B, respectively.

We define the expected values $E_A(f)$, $E_B(f)$ and $E(f)$ with respect to the probability measures q_A and q_B as follows:

$$E_A(f) = \int_{I_A} f(z,w)\mathrm{d}q_A(z); \quad E_B(f) = \int_{I_B} f(z,w)\mathrm{d}q_B(w)$$

and

$$E(f) = \int_{I_A}\int_{I_B} f(z,w)\mathrm{d}q_B(w)\mathrm{d}q_A(z).$$

Let $c_A : I_A \to \mathbb{R}_0^+$ and $c_B : I_B \to \mathbb{R}_0^+$ be measurable functions where $c_A^z = c_A(z)$ denotes the production cost of firm A when the type of firm A is $z \in I_A$ and $c_B^w = c_B(w)$ denotes the production cost of firm B when the type of firm B is $w \in I_B$. Furthermore, we assume that the expected values of c_A and c_B are finite

$$E(c_A) = E_A(c_A) = \int_{I_A} c_A^z\, \mathrm{d}q_A(z) < \infty; \quad E(c_B) = E_B(c_B) = \int_{I_B} c_B^w\, \mathrm{d}q_B(w) < \infty.$$

We assume that $\mathrm{d}q_A(z)$ denotes the probability of the *belief* of the firm B on the production costs of the firm A to be c_A^z. Similarly, assume that $\mathrm{d}q_B(w)$ denotes the probability of the belief of the firm A on the production costs of the firm B to be c_B^w.

The simplicity of the following cost deviation formulas is crucial to express the main results of this article in a clear and understandable way. The *cost deviations* of firm A and firm B

$$\Delta_A : I_A \to \mathbb{R}_0^+ \quad \text{and} \quad \Delta_B : I_B \to \mathbb{R}_0^+$$

are given respectively by $\Delta_A(z) = c_A^z - E(c_A)$ and $\Delta_B(w) = c_B^w - E(c_B)$. The *cost deviation* between the firms

$$\Delta_C : I_A \times I_B \to \mathbb{R}_0^+$$

is given by $\Delta_C(z, w) = c_A^z - c_B^w$. Since the meaning is clear, we will use through the paper the following simplified notation:

$$\Delta_A = \Delta_A(z); \quad \Delta_B = \Delta_B(w) \text{ and } \Delta_C = \Delta_C(z, w).$$

The *expected cost deviation* Δ_E between the firms is given by $\Delta_E = E(c_A) - E(c_B)$. Hence,

$$\Delta_C - \Delta_E = \Delta_A - \Delta_B.$$

Let V_A and V_B be the variances of the production costs c_A and c_B, respectively. We observe that

$$E(\Delta_C) = \Delta_E; \quad E(\Delta_A^2) = E_A(\Delta_A^2) = V_A; \quad E(\Delta_B^2) = E_B(\Delta_B^2) = V_B.$$

Furthermore,

$$\begin{aligned} E_A(\Delta_C^2) &= \Delta_B^2 + V_A + \Delta_E(\Delta_E - 2\Delta_B); \\ E_B(\Delta_C^2) &= \Delta_A^2 + V_B + \Delta_E(\Delta_E + 2\Delta_A); \\ E(\Delta_C^2) &= \Delta_E^2 + V_A + V_B. \end{aligned} \tag{33.12}$$

33.4 Complete Information

Let us consider the case where the productions costs are revealed to both firms before they choose the prices. In this case, the competition between the firms is under complete information.

A *price strategy* (p_A^{CI}, p_B^{CI}) is given by a pair of functions $p_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ and $p_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ where $p_A^{CI}(z, w)$ denotes the price of firm A and $p_B^{CI}(z, w)$ denotes the price of firm B when when the type of firm A is $z \in I_A$ and the type of firm B is $w \in I_B$.

Definition 33.2. The Hotelling model satisfies the *bounded costs (BC)* condition, if

$$|\Delta_C| < 3tl.$$

Under the BC condition, by Eqs. (33.6) and (33.7), the Nash price strategy $(\underline{p}_A^{CI}, \underline{p}_B^{CI})$ is given by

$$\underline{p}_A^{CI}(z, w) = tl + c_B + \frac{2}{3}(\Delta_C)$$

and

$$\underline{p}_B^{CI}(z, w) = tl + c_A - \frac{2}{3}(\Delta_C).$$

By Eq. (33.8), the profit $\underline{\pi}_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm A is given by

$$\underline{\pi}_A^{CI}(z, w) = \frac{(3tl - \Delta_C)^2}{18t}.$$

Similarly, by Eq. (33.9), the profit $\underline{\pi}_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm B is given by

$$\underline{\pi}_B^{CI}(z, w) = \frac{(3tl + \Delta_C)^2}{18t}.$$

The expected profit $E_B(\underline{\pi}_A^{CI})$ for firm A is given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{(6tl - 2\Delta_A - 2\Delta_E)^2 + 4V_B}{72t}$$

Similarly, the expected profit $E_A(\underline{\pi}_B^{CI})$ for firm B is given by

$$E_A(\underline{\pi}_B^{CI}) = \frac{(6tl - 2\Delta_B + 2\Delta_E)^2 + 4V_A}{72t}$$

The expected profit $E(\underline{\pi}_A^{CI})$ for firm A is given by

$$E(\underline{\pi}_A^{CI}) = \frac{(3tl - \Delta_E)^2 + V_A + V_B}{18t}.$$

Similarly, the expected profit $E(\underline{\pi}_B^{CI})$ for firm B is given by

$$E(\underline{\pi}_B^{CI}) = \frac{(3tl + \Delta_E)^2 + V_A + V_B}{18t}$$

By Eq. (33.10), the consumer surplus is given by

$$\underline{CS}^{CI}(z, w) = v_T l - \frac{3}{2}tl^2 - \frac{E(c_A) + 2E(c_B) + \Delta_A + 2\Delta_B}{3} l + \frac{(3tl - \Delta_C)^2}{36t}.$$

The expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = \upsilon_T l - \frac{3}{2}tl^2 - \frac{E(c_A) + 2E(c_B)}{3}l + \frac{(3tl - \Delta_E)^2 + V_A + V_B}{36t}.$$

By Eq. (33.11), the welfare is given by

$$\underline{W}^{CI}(z,w) = \upsilon_T l - \frac{1}{4}tl^2 - \frac{E(c_A) + E(c_B) + \Delta_A + \Delta_B}{2}l + \frac{5\Delta_C^2}{36t}.$$

Using equality (33.12), we obtain that the expected value of the welfare $E(\underline{W}^{CI})$ is given by

$$E(\underline{W}^{CI}(z,w)) = \upsilon_T l - \frac{1}{4}tl^2 - \frac{E(c_A) + E(c_B)}{2}l + \frac{5(V_A + V_B + \Delta_E^2)}{36t}.$$

33.5 Incomplete Information

Let us consider the case where the productions costs of each firm is not revealed to the other firm. In this case, the competition between the firms is under incomplete information.

A *price strategy* (p_A, p_B) is given by a pair of functions $p_A : I_A \to \mathbb{R}_0^+$ and $p_B : I_B \to \mathbb{R}_0^+$ where $p_A^z = p_A(z)$ denotes the price of firm A when the type of firm A is $z \in I_A$ and $p_B^w = p_B(w)$ denotes the price of firm B when the type of firm B is $w \in I_B$. We note that $E(p_A) = E_A(p_A)$ and $E(p_B) = E_B(p_B)$. The *indifferent consumer* $x : I_A \times I_B \to (0,l)$ is given by

$$x^{z,w} = \frac{p_B^w - p_A^z + tl}{2t}.$$

The *ex-post profits* $\pi_A^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ and $\pi_B^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EP}(z,w) = \pi_A(z,w) = (p_A^z - c_A^z)x^{z,w}$$

and

$$\pi_B^{EP}(z,w) = \pi_B(z,w) = (p_B^w - c_B^w)(l - x^{z,w}).$$

The *ex-ante profits* $\pi_A^{EA} : I_A \to \mathbb{R}_0^+$ and $\pi_B^{EA} : I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EA}(z) = E_B(\pi_A^{EP}) \quad and \quad \pi_B^{EA}(w) = E_A(\pi_B^{EP}).$$

We note that, the *expected profit* $E(\pi_A^{EP})$ of firm A is equal to $E_A(\pi_A^{EA})$ and the *expected profit* $E(\pi_B^{EP})$ of firm B is equal to $E_B(\pi_B^{EA})$.

A price strategy $(\underline{p}_A, \underline{p}_B)$ for both firms is a *Bayesian-Nash*, if for every $z \in I_A$ and for every deviation of the price $\underline{p}_A^z$ the ex-ante profit $\pi_A^{EA}(z)$ of firm A decreases, and for every $w \in I_B$ and for every deviation of the price $\underline{p}_B^w$ the ex-ante profit $\pi_B^{EA}(w)$ of firm B decreases.

For $i \in \{A, B\}$, we define

$$c_i^m = \min_{z \in I_i}\{c_i^z\} \quad \text{and} \quad c_i^M = \max_{z \in I_i}\{c_i^z\}.$$

Definition 33.3. The Hotelling model satisfies the *bounded uncertain costs (BUC)* condition, if

$$|3(c_A^z - c_B^w) + E(c_B) - E(c_A)| = |3\Delta_C - \Delta_E| < 6tl,$$

for all $z \in I_A$ and for all $w \in I_B$;

$$3\left(c_A^M + c_B^M - 2c_A^m\right) + E(c_A) - E(c_B) \leqslant 3tl + \frac{\left(E(c_A) + 2E(c_B) - 3c_A^M\right)^2}{12tl};$$

and

$$3\left(c_A^M + c_B^M - 2c_B^m\right) + E(c_B) - E(c_A) \leqslant 3tl + \frac{\left(2E(c_A) + E(c_B) - 3c_B^M\right)^2}{12tl}.$$

Let

$$\overline{\Delta} = \max_{i,j \in \{A,B\}}\{c_i^M - c_j^m\}$$

Thus, the bounded uncertain costs condition *BUC* is implied by the following stronger *SBUC* condition.

Definition 33.4. The Hotelling model satisfies the *bounded uncertain costs (SBUC)* condition, if

$$7\overline{\Delta} < 3tl$$

Theorem 33.1. *Under the BUC condition, there is a Bayesian-Nash price strategy* $(\underline{p}_A, \underline{p}_B)$. *Furthermore, the expected prices of the Bayesian-Nash price strategy are*

$$E(\underline{p}_A) = tl + E(c_A) - \frac{\Delta_E}{3};$$

$$E(\underline{p}_B) = tl + E(c_B) + \frac{\Delta_E}{3}.$$

and the Bayesian-Nash price strategy is

$$\underline{p}_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \quad \underline{p}_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}. \tag{33.13}$$

The pair of prices $(\underline{p}_A, \underline{p}_B)$ satisfies $\underline{p}_A^z > c_A^z$ and $\underline{p}_B^w > c_B^w$. Furthermore, for different production costs, the differences between the optimal prices of a firm are proportional to the differences of the production costs

$$\underline{p}_A^{z_1} - \underline{p}_A^{z_2} = \frac{c_A^{z_1} - c_A^{z_2}}{2}.$$

and

$$\underline{p}_B^{w_1} - \underline{p}_B^{w_2} = \frac{c_B^{w_1} - c_B^{w_2}}{2}.$$

for all $z_1, z_2 \in I_A$ and $w_1, w_2 \in I_B$. Hence, half of the production costs value is incorporated in the price.

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firm. Hence it is the main economic information for both firms. By Eq. (33.13), the ex-post profit of firm A is

$$\underline{\pi}_A^{EP}(z, w) = \frac{(6tl + \Delta_E - 3\Delta_C)(6tl + \Delta_E - 3\Delta_C - 3\Delta_B)}{72t}$$

and the ex-post profit of firm B is

$$\underline{\pi}_B^{EP}(z, w) = \frac{(6tl - \Delta_E + 3\Delta_C)(6tl - \Delta_E + 3\Delta_C - 3\Delta_A)}{72t}.$$

We observe that the differences between the ex-post profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

$$\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_B^{EP}(z, w) = \frac{6tl(\Delta_A - \Delta_B) + (\Delta_E - 3\Delta_C)(8tl - \Delta_A - \Delta_B)}{24t}$$

Furthermore, for different production costs, the differences between the ex-post profit of a firm is given by

$$\underline{\pi}_A^{EP}(z_1, w) - \underline{\pi}_A^{EP}(z_2, w) = \frac{(c_A^{z_2} - c_A^{z_1})(12tl - \Delta_E + 3(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24t}$$

and

$$\underline{\pi}_B^{EP}(z, w_1) - \underline{\pi}_B^{EP}(z, w_2) = \frac{(c_B^{w_2} - c_B^{w_1})(12tl + \Delta_E + 3(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24t}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The ex-ante profit of a firm is the expected profit of the firm that knows its production cost but is uncertain about the production costs of the competitor firm. The ex-ante profit of firm A is

$$\underline{\pi}_A^{EA}(z) = \frac{(6tl - 3\Delta_A - 2\Delta_E)^2}{72t}.$$

Similarly, the ex-ante profit of firm B is

$$\underline{\pi}_B^{EA}(w) = \frac{(6tl - 3\Delta_B + 2\Delta_E)^2}{72t}.$$

We observe that the differences between the ex-ante profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4tl - \Delta_A - \Delta_B)(3(\Delta_B - \Delta_A) - 4\Delta_E)}{24t}$$

Furthermore, for different production costs, the differences between the ex-ante profits of a firm are given by

$$\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2) = \frac{(c_A^{z_2} - c_A^{z_1})(3(4tl + 2E(c_A) - c_A^{z_1} - c_A^{z_2}) - 4\Delta_E)}{24t}$$

and

$$\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2) = \frac{(c_B^{w_2} - c_B^{w_1})(3(4tl + 2E(c_B) - c_B^{w_1} - c_B^{w_2}) + 4\Delta_E)}{24t}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

We observe that the ex-ante and the ex-posts profits of both firms are strictly positive with respect to the Bayesian-Nash price strategy determined in Theorem 33.1. Hence, the expected profits of both firms are also strictly positive.

The expected profit of the firm is the expected gain of the firm. The expected profit of firm A is given by

$$E(\underline{\pi}_A^{EP}) = \frac{(3tl - \Delta_E)^2}{18t} + \frac{V_A}{8t}$$

Similarly, the expected profit of firm B is given by

$$E(\underline{\pi}_B^{EP}) = \frac{(3tl + \Delta_E)^2}{18t} + \frac{V_B}{8t}.$$

The ex-post consumer surplus is the realized gain of the consumers community for given outcomes of the production costs of both firms. Under incomplete information, by Eq. (33.4), the ex-post consumer surplus is

$$\underline{CS}^{EP} = v_T l - \frac{3}{2}tl^2 - \frac{l}{3}(2E(c_B) + E(c_A)) - \frac{\Delta_B l}{2} + \frac{(6tl - 3\Delta_C + \Delta_E)^2}{144t}.$$

The expected value of the consumer surplus is the expected gain of the consumers community for all possible outcomes of the production costs of both firms. The expected value of the consumer surplus $E(\underline{CS}^{EP})$ is given by

$$E(\underline{CS}^{EP}) = v_T l - \frac{3}{2}tl^2 - \frac{l}{3}(2E(c_B) + E(c_A)) + \frac{(6tl - 2\Delta_E)^2 + 9(V_A + V_B)}{144t}.$$

The ex-post welfare is the realized gain of the state that includes the gains of the consumers community and the gains of the firms for a given outcomes of the production costs of both firms. By Eq. (33.5), the ex-post welfare is

$$\underline{W}^{EP} = v_T l - \frac{1}{4}tl^2 - \frac{E(c_A) + E(c_B) + \Delta_A + \Delta_B}{2} - \frac{3\Delta_C(2\Delta_E - 9\Delta_C) + (\Delta_E)^2}{144t}.$$

The expected value of the welfare is the expected gain of the state for all possible outcomes of the production costs of both firms. The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$E(\underline{W}^{EP}) = v_T l - \frac{1}{4}tl^2 - \frac{E(c_A) + E(c_B)}{2} + \frac{27(V_A + V_B) + 20\Delta_E^2}{144t}$$

33.6 Incomplete Versus Complete Information

Corollary 33.1. *The difference between the ex-post profit and the profit, under complete information, for firm A is*

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w) = \frac{(\Delta_A - \Delta_B)(\Delta_A + 2\Delta_B) - 2(3tl - \Delta_C)(2\Delta_A + \Delta_B)}{72t}.$$

The difference between the ex-post profit and the profit, under complete information, for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w) = \frac{(\Delta_B - \Delta_A)(\Delta_B + 2\Delta_A) - 2(3tl + \Delta_C)(2\Delta_B + \Delta_A)}{72t}.$$

Corollary 33.2. *The difference between the ex-ante profit $E_B(\underline{\pi}_A^{EP})$ and $E_B(\underline{\pi}_A^{CI})$ for firm A is*

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\,(5\Delta_A - 4(3tl - \Delta_E))}{72t} - \frac{V_B}{18t}.$$

The difference between the ex-ante profit $E_A(\underline{\pi}_B^{EP})$ and $E_A(\underline{\pi}_B^{CI})$ for firm B is

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\,(5\Delta_B - 4(3tl + \Delta_E))}{72t} - \frac{V_A}{18t}.$$

The difference between the expected profits of firm A with complete and incomplete information is given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = \frac{5V_A - 4V_B}{72t}. \tag{33.14}$$

The difference between the expected profits of firm B with complete and incomplete information is given by

$$E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{5V_B - 4V_A}{72t}. \tag{33.15}$$

Corollary 33.3. *The difference between the ex-post consumer surplus and the consumer surplus, under complete information, is*

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{(\Delta_A + \Delta_B)l}{4} + \frac{(\Delta_B - \Delta_A)(\Delta_B - \Delta_A - 4\Delta_C)}{144t}. \tag{33.16}$$

Therefore, Eq. (33.16) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of consumer surplus $\underline{CS}^{EP} > \underline{CS}^{CI}$.

The difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{5(V_A + V_B)}{144t}. \tag{33.17}$$

Therefore, in expected value the consumer surplus is greater with incomplete information than with complete information.

The difference between the ex-post welfare and the welfare, under complete information, is

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{7(\Delta_C)^2 - 6\Delta_C \Delta_E - (\Delta_E)^2}{144t}. \tag{33.18}$$

Therefore, Eq. (33.18) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of welfare $\underline{W}^{EP} > \underline{W}^{CI}$.

The difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7(V_A + V_B)}{144t}. \tag{33.19}$$

Therefore, in expected value the welfare is greater with incomplete information than with complete information.

33.7 Symmetric Hotelling

A Hotelling game is *symmetric*, if $(I_A, \Omega_A, q_A) = (I_B, \Omega_B, q_B)$ and $c = c_A = c_B$. Hence, we observe that all the formulas of this paper hold with the following simplifications $\Delta_E = 0$, $E(c) = E(c_A) = E(c_B)$ and $V = V_A = V_B$. Let $c^M = c_A^M = c_B^M$ and $c^m = c_A^m = c_B^m$. The bounded uncertain costs in the symmetric case can be written in the following way

Definition 33.5. The symmetric Hotelling model satisfies the *bounded uncertain costs (BUC)* condition, if

$$|\Delta_C| < 2tl,$$

for all $z \in I_A$ and for all $w \in I_B$;

$$2\left(c^M - c^m\right) \leqslant tl + \frac{\left(c^M - E(c)\right)^2}{4tl};$$

Thus, the bounded uncertain costs condition *BUC* is implied by the following stronger condition.

Definition 33.6. The symmetric Hotelling model satisfies the *bounded uncertain costs (SBUC)* condition, if

$$2\overline{\Delta} < tl$$

Under the *BUC* condition, the expected prices of Bayesian-Nash price strategy are given by

$$E(\underline{p}_A) = E(\underline{p}_B) = tl + E(c).$$

We observe that the difference between the expected profits of firm A and B with complete and incomplete information is positive and it is given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{V}{72t}.$$

Furthermore, the difference between the welfare and the welfare with complete and incomplete information is positive and it is given by

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{7(\Delta_C)^2}{144t}.$$

33.8 Conclusion

Under complete and incomplete information, we studied the economic impact of the Nash price strategies in the profits of the firms, consumer surplus and welfare. Under incomplete information, we observed that the Bayesian-Nash price strategies do not depend upon the distributions of the production costs of the firms, except on their first moments. Furthermore, the prices of each firm, at equilibrium, are affine with respect to the expected costs of both firms and to their own costs. The corresponding expected profits are quadratic in the expected cost of both firms, in its own cost and in the transportation cost. We computed the ex-post consumer surplus and the ex-post welfare and we explicitly determined in which cases it is better to have uncertainty in the production costs instead of complete information in terms of profits, consumer surplus and in terms of welfare.

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