Chapter 1
Renormalization of Circle Diffeomorphism Sequences and Markov Sequences

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Abstract We show a one-to-one correspondence between circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization and smooth Markov sequences.

1.1 Introduction

Following [2–7, 15, 19–23], we present the concept of renormalization applied to circle diffeomorphism sequences. These concepts are essential for extending the results presented in [19, 20] to all Anosov diffeomorphisms on surfaces, i.e., for proving a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms and pairs of C^{1+} circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization (see also [1, 18–20]). The main point in this paper is to establish the existence of a one-to-one correspondence between C^{1+} circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization and smooth Markov sequences. This correspondence is a key step in passing from

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circle diffeomorphisms to Anosov diffeomorphisms because the Markov sequences encode the smooth information of the expanding and contracting laminations of the Anosov diffeomorphisms [8–14, 16, 17].

1.2 Circle Diffeomorphisms

Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be a sequence of positive integers and let $\gamma(\mathbf{a}) = 1/(a_0 + 1/(a_1 + 1/\cdots))$. For every $i \in \mathbb{N}_0$, let $\gamma_i = \gamma_i(\mathbf{a}) = 1/(a_i + 1/(a_{i+1} + 1/\cdots))$ and let $\mathbb{S}_i$ be a *counterclockwise oriented circle* homeomorphic to the circle $\underline{\mathbb{S}}_i = \mathbb{R}/(1+\gamma_i)\mathbb{Z}$.

An *arc* in $\mathbb{S}_i$ is the image of a nontrivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to \mathbb{S}_i$. If I is closed (respectively open), we say that $\alpha(I)$ is a *closed* (respectively *open*) *arc* in $\mathbb{S}_i$. We denote by (a,b) (respectively $[a,b]$) the positively oriented open (respectively closed) arc on $\mathbb{S}_i$ beginning at the point $a \in \mathbb{S}_i$ and ending at the point $b \in \mathbb{S}_i$. A C^{1+} atlas $\mathscr{A}_i$ in $\mathbb{S}_i$ is a set of charts such that (1) every small arc of $\mathbb{S}_i$ is contained in the domain of some chart in $\mathscr{A}_i$, and (2) the overlap maps are $C^{1+\alpha}$ compatible, for some $\alpha > 0$.

Let $\underline{\mathscr{A}}_i$ denote the affine atlas whose charts are isometries with respect to the usual norm in $\underline{\mathbb{S}}_i$. Let the *rigid rotation* $\underline{g}_i : \underline{\mathbb{S}}_i \to \underline{\mathbb{S}}_i$ be the affine homeomorphism with respect to the atlas $\underline{\mathscr{A}}_i$ with rotation number $\gamma_i/(1+\gamma_i)$.

A homeomorphism $h : \underline{\mathbb{S}}_i \to \mathbb{S}_i$ is *quasisymmetric* if there exists a constant $C > 1$ such that for each two arcs I_1 and I_2 of $\underline{\mathbb{S}}_i$ with a common endpoint and such that $|I_1|_i = |I_2|_i$, we have $|h(I_1)|_i/|h(I_2)|_i < C$, where the lengths are measured in the charts of $\underline{\mathscr{A}}_i$ and $\mathscr{A}_i$.

A C^{1+} *circle diffeomorphism sequence* $(g_i, \mathbb{S}_i, \mathscr{A}_i)_{i=0}^{\infty}$ is a sequence of triples $(g_i, \mathbb{S}_i, \mathscr{A}_i)$ with the following properties: (1) $g_i : \mathbb{S}_i \to \mathbb{S}_i$ is a $C^{1+\alpha}$ diffeomorphism with respect to the $C^{1+\alpha}$ atlas $\mathscr{A}_i$ for some $\alpha > 0$; and (2) g_i is quasisymmetric conjugate to the rigid rotation $\underline{g}_i$ with respect to the atlas $\underline{\mathscr{A}}_i$.

We denote the C^{1+} circle diffeomorphism $(g_i, \mathbb{S}_i, \mathscr{A}_i)$ by g_i. In particular, we denote the rigid rotation $(\underline{g}_i, \underline{\mathbb{S}}_i, \underline{\mathscr{A}}_i)$ by $\underline{g}_i$.

1.2.1 Horocycles

Let us mark a point in $\mathbb{S}_i$, which we will denote by $0_i \in \mathbb{S}_i$. Let $S_i^0 = [0_i, g_i(0_i)]$ be the oriented closed arc in $\mathbb{S}_i$ with endpoints 0_i and $g_i(0_i)$. For every $k \in \{0, \ldots, a_i\}$, let $S_i^k = \left[g_i^k(0_i), g_i^{k+1}(0_i)\right]$ be the oriented closed arc in $\mathbb{S}_i$ with endpoints $g_i^k(0_i)$ and $g_i^{k+1}(0_i)$ and such that $S_i^k \cap S_i^{k-1} = \{g_i^k(0)\}$. Let $S_i^{a_i+1} = [g_i^{a_i+1}(0_i), 0_i]$ be the oriented closed arc in $\mathbb{S}_i$ with endpoints $g_i^{a_i+1}(0_i)$ and 0_i.

We introduce an *equivalence relation* $\sim$ in $\mathbb{S}_i$ by identifying the $a_i + 1$ points $g_i(0), \ldots, g_i^{a_i+1}(0)$ and form the topological space $H_i(\mathbb{S}_i, g_i) = \mathbb{S}_i/\sim$. We take the orientation in H_i as the reverse orientation of the one induced by $\mathbb{S}_i$. We call this oriented topological space the *horocycle*, and we denote it by $H_i = H_i(\mathbb{S}_i, g_i)$.

We consider the quotient topology in H_i. Let $\pi_{g_i} : \mathbb{S}_i \to H_i$ be the natural projection. The point

$$\xi_i = \pi_{g_i}(g_i(0_i)) = \cdots = \pi_{g_i}(g_i^{a_i+1}(0_i)) \in H_i$$

is called the *junction* of the horocycle H_i. For every $k \in \{0,\ldots,a_i\}$, let $S^k_{i,H} = S^k_{i,H}(\mathbb{S}_i, g_i) \subset H_i$ be the projection by π_{g_i} of the closed arc S^k_i. Let $R_i\mathbb{S}_i = S^0_{i,H} \cup S^{a+1}_{i,H}$ be the *renormalized circle* in H_i. The horocycle H_i is the union of the renormalized circle $R_i\mathbb{S}_i$ with the circles $S^k_{i,H}$ for every $k \in \{1,\ldots,a_i\}$.

A *parameterization* in H_i is the image of a nontrivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to H_i$. If I is closed (respectively open), we say that $\alpha(I)$ is a *closed* (respectively *open*) *arc* in H_i. A *chart* in H_i is the inverse of a parameterization. A *topological atlas* $\mathscr{B}$ on the horocycle H_i is a set of charts $\{(j,J)\}$ on H_i with the property that every small arc is contained in the domain of a chart in $\mathscr{B}$, i.e., for every open arc K in H_i and $x \in K$, there exists a chart $\{(j,J)\} \in \mathscr{B}$ such that $J \cap K$ is a nontrivial open arc in H_i and $x \in J \cap K$. A C^{1+} *atlas* $\mathscr{B}$ *in* H_i is a topological atlas $\mathscr{B}$ such that the overlap maps are $C^{1+\alpha}$ and have $C^{1+\alpha}$ uniformly bounded norms, for some $\alpha > 0$.

Let $\mathscr{A}_i$ be a C^{1+} atlas on $\mathbb{S}_i$ in which $g_i : \mathbb{S}_i \to \mathbb{S}_i$ is a C^{1+} circle diffeomorphism. We are going to construct a C^{1+} atlas $\mathscr{A}_i^H$ on H_i that we call the *extended pushforward* $\mathscr{A}_i^H = (\pi_{g_i})_* \mathscr{A}_i$ of the atlas $\mathscr{A}_i$ on $\mathbb{S}_i$. If $x \in H_i \backslash \{\xi_i\}$, then there exists a sufficiently small open arc $J \subset H_i$ containing x such that $\pi_{g_i}^{-1}(J)$ is contained in the domain of some chart $(I,\hat{\imath})$ of $\mathscr{A}_i$. In this case, we define $(J, \hat{\imath} \circ \pi_{g_i}^{-1})$ as a chart in $\mathscr{A}_i^H$. If $x = \xi_i$ and J is a small arc containing ξ_i, then either (i) $\pi_{g_i}^{-1}(J)$ is an arc in $\mathbb{S}_i$ or (ii) $\pi_{g_i}^{-1}(J)$ is a disconnected set that is the union of two connected components.

In case (i), $\pi_{g_i}^{-1}(J)$ is connected, and it is contained in the domain of some chart $(I,\hat{\imath}) \in \mathscr{A}_i$. Therefore, we define $\left(J, \hat{\imath} \circ \pi_{g_i}^{-1}\right)$ as a chart in $\mathscr{A}_i^H$.

In case (ii), $\pi_{g_i}^{-1}(J)$ is a disconnected set that is the union of two connected arcs I^L_l and I^R_r of the form $I^L_l = (c^L_l, g^l_i(0)]$ and $I^R_r = [g^r_i(0), c^R_r)$, respectively, for all $l, r \in \{1,\ldots,a_i+1\}$. Let J^L_l and J^R_r be the arcs in H_i defined by $J^L_l = \pi_{g_i}(I^L_l)$ and $\pi_{g_i}(I^R_r)$ respectively. Then $J = J^L_l \cup J^R_r$ is an arc in H_i with the property that $J^L_l \cap J^R_r = \{\xi_i\}$, for every $l, r \in \{1,\ldots,a_i+1\}$. We call such an arc J an (l,r)-*arc*, and we denote it by $J_{l,r}$. Let $j_{l,r} : J_{l,r} \to \mathbb{R}$ be defined by

$$j_{l,r}(x) = \begin{cases} \hat{\imath} \circ \pi_{g_i}^{-1}(x) & \text{if } x \in J^R_r, \\ \hat{\imath} \circ g_i^{r-l} \circ \pi_{g_i}^{-1}(x) & \text{if } x \in J^L_l. \end{cases}$$

Let $(I,\hat{\imath}) \in \mathscr{A}_i$ be a chart such that $\pi_{g_i}(I) \supset J_{l,r}$. Then we define $(J_{l,r}, j_{l,r})$ as a chart in $\mathscr{A}_i^H$ (see Fig. 1.1). We call the atlas determined by these charts the *extended pushforward atlas of* $\mathscr{A}_i$, and by abuse of notation, we will denote it by $\mathscr{A}_i^H = (\pi_{g_i})_* \mathscr{A}_i$.

Let the marked point $\underline{0}_i$ in $\underline{\mathbb{S}}_i$ be the natural projection of $0 \in \mathbb{R}$ onto $\underline{0}_i \in \underline{\mathbb{S}}_i = \mathbb{R}/(1+\gamma_i)\mathbb{Z}$. Let $\underline{S}^0_i = [0_i, \underline{g}_i(0_i)]$ and $\underline{S}^k_i = [\underline{g}^k_i(0_i), \underline{g}^{k+1}_i(0_i)]$. Furthermore, let

$$\underline{H}_i = H_i(\underline{\mathbb{S}}_i, \underline{g}_i)\ ,\ \underline{S}^k_{i,H} = S^k_{i,H}(\underline{\mathbb{S}}_i, \underline{g}_i)\ ,\ R_i\underline{\mathbb{S}}_i = \underline{\mathbb{S}}^0_{i,H} \cup \underline{\mathbb{S}}^{a+1}_{i,H} \text{ and } \underline{\mathscr{A}}^{H_i} = \left(\pi_{\underline{g}_i}\right)_* \underline{\mathscr{A}}_i.$$

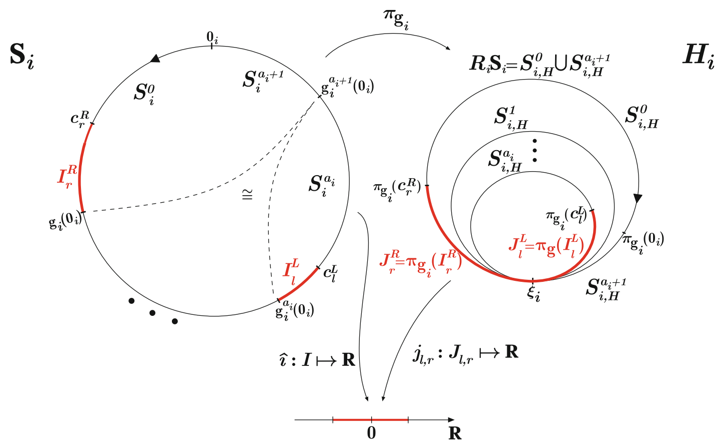

Fig. 1.1 The horocycle H_i and the chart $j_{l,r} : J_{l,r} \to \mathbb{R}$ in case (ii). The junction ξ_i of the horocycle is equal to $\xi_i = \pi_{g_i}(g_i(0_i)) = \cdots = \pi_{g_i}(g_i^{a_i}(0_i)) = \pi_{g_i}(g_i^{a_i+1}(0_i))$

1.3 Renormalization

The *renormalization of a C^{1+} circle diffeomorphism* g_i is the triple $(R_i g_i, R_i\mathbb{S}_i, R_i\mathscr{A}_i)$, where (1) $R_i\mathbb{S}_i$ is the renormalized circle with the orientation of the horocycle H_i, i.e., the reversed orientation of the orientation induced by $\mathbb{S}_i$; (2) the *renormalized atlas* $R_i\mathscr{A}_i = \mathscr{A}_i^H|_{R_i\mathbb{S}_i}$ is the set of all charts in $\mathscr{A}_i^H$ with domains contained in $R_i\mathbb{S}_i$; and (3) $R_i g_i : R_i\mathbb{S}_i \to R_i\mathbb{S}_i$ is the continuous map given by

$$R_i g_i(x) = \begin{cases} \pi_{g_i} \circ g_i^{a_i+1} \circ \left(\pi_{g_i}|_{S^0_{i,H}}\right)^{-1}(x) & \text{if } x \in S^0_{i,H}, \\ \pi_{g_i} \circ g_i \circ \left(\pi_{g_i}|_{S^{a_i+1}_{i,H}}\right)^{-1}(x) & \text{if } x \in S^{a_i+1}_{i,H}. \end{cases}$$

We denote the C^{1+} renormalization $(R_i g_i, R_i\mathbb{S}_i, R_i\mathscr{A}_i)$ of g_i by $R_i g_i$.

By construction, the renormalization $R_i\underline{g}_i$ of the rigid rotation $\underline{g}_i$ is affine conjugate to the rigid rotation $\underline{g}_{i+1}$. Hence from now on, we identify $(R_i\underline{g}_i, R_i\underline{\mathbb{S}}_i, R_i\underline{\mathscr{A}}_i)$ with $(\underline{g}_{i+1}, \underline{\mathbb{S}}_{i+1}, \underline{\mathscr{A}}_{i+1})$.

Recall that a C^{1+} *circle diffeomorphism* $g : \mathbb{S}_i \to \mathbb{S}_i$ is a $C^{1+\alpha}$ diffeomorphism with respect to a $C^{1+\alpha}$ atlas $\mathscr{A}$ on $\mathbb{S}_i$, for some $\alpha > 0$, that is quasisymmetric conjugate to a rigid rotation $\underline{g} : \mathbb{S}_i \to \mathbb{S}_i$ with respect to an affine atlas $\underline{\mathscr{A}}$ on $\underline{\mathbb{S}}_i$.

The renormalization $R_i g_i$ is a C^{1+} circle diffeomorphism quasisymmetric conjugate to the rigid rotation $\underline{g}_{i+1}$. Hence $R_i g_i$ is quasisymmetric conjugate to the C^{1+} circle diffeomorphism g_{i+1}. The marked point $0_i \in \mathbb{S}_i$ determines the marked point $0_{R_i\mathbb{S}_i} = \pi_{g_i}(0_i)$ in the circle $R_i\mathbb{S}_i$. Thus, there is a unique topological conjugacy h_i between $R_i g_i$ and g_{i+1} such that $h_i(0_{R_i\mathbb{S}_i}) = 0_{i+1}$ (see Fig. 1.2).

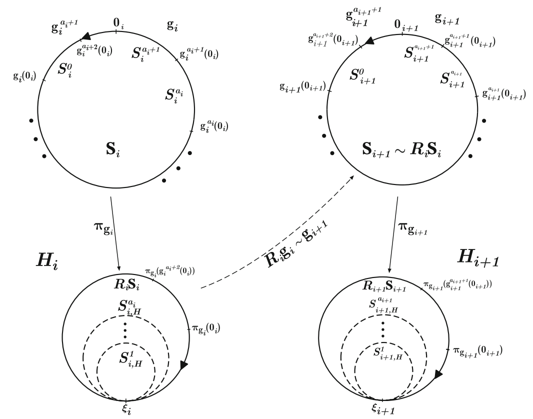

Fig. 1.2 The horocycles H_i and H_{i+1}, and the renormalized map $R_i g_i : R_i\mathbb{S}_i \to R_i\mathbb{S}_i$. Here $\xi_i = g_i(0_i) = \ldots = g_i^{a_i+1}(0_i)$, $\xi_{i+1} = g_{i+1}(0_{i+1}) = \cdots = g_{i+1}^{a_{i+1}+1}(0_{i+1}))$, and the map $R_i g_i$ is identified with g_{i+1}

A C^{1+} circle diffeomorphism g_0 determines a unique C^{1+} *renormalization circle diffeomorphism sequence* $\mathbf{R}(g_0) = (g_i, \mathbb{S}_i, \mathscr{A}_i)_{i=0}^{\infty}$ given by

$$(g_i, \mathbb{S}_i, \mathscr{A}_i) = (R_i \circ \cdots \circ R_0 g_0, R_i \circ \cdots \circ R_0 \mathbb{S}_0, R_i \circ \cdots \circ R_0 \mathscr{A}_0).$$

We note that the C^{1+} renormalization circle diffeomorphism sequence $\mathbf{R}(g_0)$ is a C^{1+} circle diffeomorphism sequence.

We say that a sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ is *n-periodic* if n is the least integer such that $a_{i+n} = a_i$ for every $i \in \mathbb{N}_0$. We observe that given a n-periodic sequence of positive integers $\mathbf{a} = (a_i)_{i=0}^{\infty}$, $\gamma_i = \gamma_i(\mathbf{a}) = 1/(a_i + 1/(a_{i+1} + 1/\cdots))$ is equal to γ_{i+n} for every $i \in \mathbb{N}_0$. Hence there exists a topological conjugacy $\phi_i : \mathbb{S}_i \to \mathbb{S}_{i+n}$ such that

$$\phi_i \circ g_i = g_{i+n} \circ \phi_i,$$

because g_i and $g_{i+n} = R_{i+n} \circ \cdots \circ R_i g_i$ are C^{1+} circle diffeomorphisms with the same rotation number $\gamma_i = \gamma_{i+n}$.

We say that a sequence $\mathbf{R}(g_0)$ is a C^{1+} *n-periodic point of renormalization* if ϕ_i is C^{1+} for every $i \in \mathbb{N}$.

1.4 Markov Maps

Let $\mathbf{R}(g_0)$ be the renormalization circle diffeomorphism sequence associated to the C^{1+} circle diffeomorphism g_0. The *Markov map* $M_i : H_i \to H_{i+1}$ is given by

$$M_i(x) = \begin{cases} \pi_{g_{i+1}}(x) & \text{if } x \in R_i\mathbb{S}_i, \\ \pi_{g_{i+1}} \circ \pi_{g_i} \circ g_i^{-k} \circ \pi_{g_i}^{-1}(x) & \text{if } x \in S^k_{i,H_i}, \text{ for } k = 1, \ldots, a_i. \end{cases}$$

The *Markov sequence* $(M_i)_{i=0}^{\infty}(g_0)$ associated to a C^{1+} circle diffeomorphism g_0 is the sequence of Markov maps $M_i : H_i \to H_{i+1}$ for $i \in \mathbb{N}_0$. Two Markov sequences $(M_i)_{i=0}^{\infty}(g_0)$ and $(N_i)_{i=0}^{\infty}(g_0)$ are *quasisymmetric conjugate* if there is a sequence $(h_i)_{i=0}^{\infty}$ of quasisymmetric maps h_i such that $M_{i+1} \circ h_i = h_{i+1} \circ M_i$ for each $i \in \mathbb{N}_0$.

The *rigid Markov sequence* $(\underline{M}_i)_{i=0}^{\infty} = (M_i)_{i=0}^{\infty}(\underline{g}_0)$ is the Markov sequence associated to the rigid rotation $\underline{g}_0$. The rigid Markov maps $\underline{M}_i : \underline{H}_i \to \underline{H}_{i+1}$ are affine with respect to the atlases $\underline{\mathscr{A}}_i^H$ and $\underline{\mathscr{A}}_{i+1}^H$ (see Fig. 1.3).

The Markov sequence $(M_i)_{i=0}^{\infty}(g_0)$ has the following properties: (i) the Markov maps M_i are local $C^{1+\alpha}$ diffeomorphisms for some $\alpha > 0$, and (ii) the Markov sequence $(M_i)_{i=0}^{\infty}(g_0)$ is quasisymmetric conjugate to the rigid Markov sequence $(\underline{M}_i)_{i=0}^{\infty}(\underline{g}_0)$ because g_i is quasisymmetric conjugate to $\underline{g}_i$.

The *n-extended Markov sequence* $(\mathbf{M}_i)_{i=0}^{n-1}(g_0)$ is the sequence of the *n-extended Markov maps* $\mathbf{M}_i(g_0) : H_i \to H_i$ defined by

$$\mathbf{M}_i(g_0) = \phi_i^{-1} \circ M_{i+n} \circ \cdots \circ M_i.$$

We observe that a sequence $\mathbf{R}(g_0)$ is a C^{1+} *n-periodic point of renormalization* if and only if the n-extended Markov maps $\mathbf{M}_i : H_i \to H_i$ are C^{1+} for every $i \in \mathbb{N}$.

The *rigid n-extended Markov sequence* $(\underline{\mathbf{M}}_i)_{i=0}^{n-1} = (\mathbf{M}_i)_{i=0}^{n-1}(\underline{g}_0)$ is the n-extended Markov sequence associated to the rigid rotation $\underline{g}_0$. The rigid n-extended Markov maps $\mathbf{M}_0, \ldots, \mathbf{M}_{n-1}$ are affine with respect to the atlases $\underline{\mathscr{A}}_0^H, \ldots, \underline{\mathscr{A}}_{n-1}^H$, respectively, because the conjugacy maps $\underline{\phi}_i : \underline{\mathbb{S}}_i \to \underline{\mathbb{S}}_{i+n}$ are affine.

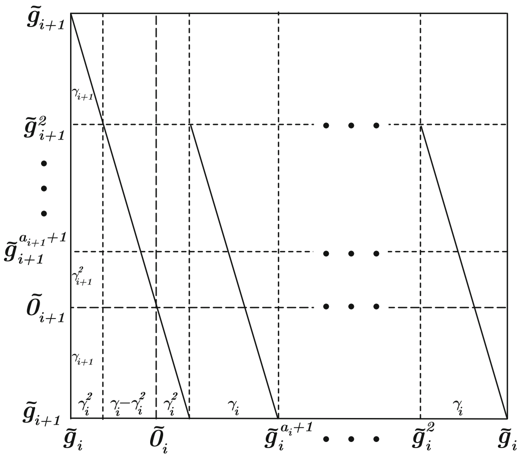

Fig. 1.3 A representation of the rigid Markov map $\underline{M}_i : \underline{H}_i \to \underline{H}_{i+1}$ with respect to the atlases $\underline{\mathscr{A}}_i^H$ and $\underline{\mathscr{A}}_{i+1}^H$, respectively. Here we represent by $\tilde{0}_i$ and $\tilde{0}_{i+1}$ the points $\pi_{\underline{g}_i}(\underline{0}_i)$ and $\pi_{\underline{g}_{i+1}}(\underline{0}_{i+1})$, respectively, and by $\tilde{g}_k^l$ the points $\pi_{\underline{g}_k} \circ \underline{g}_k^l(\underline{0}_k)$, for $k \in \{i, i+1\}$ and $l \in \{1, \ldots, a_k + 1\}$

If $\phi_i : \mathbb{S}_i \to \mathbb{S}_{i+n}$ is $C^{1+\alpha}$, then the n-extended Markov sequence $(\mathbf{M}_i)_{i=0}^{n-1}(g_0)$ has the following properties: (1) the n-extended Markov maps $\mathbf{M}_i$ are local $C^{1+\alpha}$ diffeomorphisms, for some $\alpha > 0$, because the Markov maps $M_0, \ldots M_{n-1}$ of the sequence $(M_i)_{i=0}^{n-1}(g_0)$ are local $C^{1+\alpha}$ diffeomorphisms, and (2) the n-extended Markov maps $\mathbf{M}_i$ are quasisymmetric conjugate to the rigid n-extended Markov maps $\underline{\mathbf{M}}_i$ because the Markov maps $M_0,\ldots M_{n-1}$ are quasisymmetric conjugate to $\underline{M}_0,\ldots,\underline{M}_{n-1}$.

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