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THE CIRCLE AND THE SOLENOID

A. A. Pinto

DMP, Faculdade de Ciências, Universidade do Porto
4000 Porto, Portugal

D. Sullivan

Einstein chair, Graduate Center
City University of New York and SUNY Stony Brook
New York 11794-3651, U.S.A.

Abstract. In the paper, we discuss two questions about degree d smooth expanding circle maps, with $d \geq 2$. (i) We characterize the sequences of asymptotic length ratios which occur for systems with Hölder continuous derivative. The sequences of asymptotic length ratios are precisely those given by a positive Hölder continuous function s (solenoid function) on the Cantor set C of d-adic integers satisfying a functional equation called the matching condition. In the case of the 2-adic integer Cantor set, the functional equation is

$$s(2x+1) = \frac{s(x)}{s(2x)}\left(1 + \frac{1}{s(2x-1)}\right) - 1.$$

We also present a one-to-one correspondence between solenoid functions and affine classes of exponentially fast d-adic tilings of the real line that are fixed points of the d-amalgamation operator. (ii) We calculate the precise maximum possible level of smoothness for a representative of the system, up to diffeomorphic conjugacy, in terms of the functions s and $cr(x) = (1 + s(x))/(1 + (s(x+1))^{-1})$. For example, in the Lipschitz structure on C determined by s, the maximum smoothness is $C^{1+\alpha}$ for $0 < \alpha \leq 1$ if and only if s is α-Hölder continuous. The maximum smoothness is $C^{2+\alpha}$ for $0 < \alpha \leq 1$ if and only if cr is $(1+\alpha)$-Hölder. A curious connection with Mostow type rigidity is provided by the fact that s must be constant if it is α-Hölder for $\alpha > 1$.

1. **Introduction.** One could say that this paper is about the space $A(2)$ of sequences $\{a_1, a_2, \ldots\}$ of positive real numbers satisfying the following:

(i) a_n/a_m is exponentially near 1 if $n - m$ is divisible by a high power of two, and
(ii) $a_3, a_5, a_7, \ldots$ is constructed from a_1 and $a_2, a_4, a_6, \ldots$ by the recursion

$$a_{2n+1} = \frac{a_n}{a_{2n}}\left(1 + \frac{1}{a_{2n-1}}\right) - 1. \tag{1}$$

The only explicit element in $A(2)$ that we know is $\{1, 1, 1, \ldots\}$.

Theorem 1. *The set $A(2)$ is canonically isomorphic to*

(i) the set of all possible affine structures on the leaves of the dyadic solenoid $\tilde{S}(2)$ that are transversely Hölder continuous and invariant by the natural dynamics $\tilde{E}(2) : \tilde{S}(2) \to \tilde{S}(2)$;

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