GENERATING THE ALGEBRAIC THEORY OF $C(X)$: THE CASE OF PARTIALLY ORDERED COMPACT SPACES

DIRK HOFMANN, RENATO NEVES, AND PEDRO NORA

ABSTRACT. It is known since the late 1960's that the dual of the category of compact Hausdorff spaces and continuous maps is a variety – not finitary, but bounded by $\aleph_1$. In this note we show that the dual of the category of partially ordered compact spaces and monotone continuous maps is a $\aleph_1$-ary quasivariety, and describe partially its algebraic theory. Based on this description, we extend these results to categories of Vietoris coalgebras and homomorphisms. We also characterise the $\aleph_1$-copresentable partially ordered compact spaces.

1. INTRODUCTION

The motivation for this paper stems from two very different sources. Firstly, it is known since the end of the 1960's that the dual of the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps is a variety – not finitary, but bounded by $\aleph_1$. More in detail,

- in [Dus69] it is proved that the representable functor $\hom(-,[0,1])\colon \mathsf{CompHaus}^{\mathrm{op}} \to \mathsf{Set}$ is monadic,
- the unit interval $[0,1]$ is shown to be a $\aleph_0$-copresentable compact Hausdorff space in [GU71],
- a presentation of the algebra operations of $\mathsf{CompHaus}^{\mathrm{op}}$ is given in [Isb82], and
- a complete description of the algebraic theory of $\mathsf{CompHaus}^{\mathrm{op}}$ is obtained in [MR17].

It is also worth mentioning that, by the famous Gelfand duality theorem [Gel41], $\mathsf{CompHaus}$ is dually equivalent to the category of commutative C^*-algebras and homomorphisms; the algebraic theory of (commutative) C^*-algebras is extensively studied in [Neg71, PR89, PR93]. Our second source of inspiration is the theory of coalgebras. In [KKV04] the authors argue that the category BooSp of Boolean spaces and continuous maps "is an interesting base category for coalgebras"; among other reasons, due to the connection with modal logic. A similar study based on the Vietoris functor on the category Priest of Priestley spaces and monotone continuous maps can be found in [CLP91, Pet96, BKR07]. Arguably, the categories BooSp and Priest are very suitable in this context because they are duals of finitary varieties (due to the famous Stone dualities [Sto36, Sto38a, Sto38b]), a property which extends to categories of coalgebras and therefore guarantees for instance good completeness properties.

In this note we go a step further and study the category $\mathsf{PosComp}$ of partially ordered compact spaces and monotone continuous maps, which was introduced in [Nac50] and constitutes a natural extension of both the category $\mathsf{CompHaus}$ and the category Priest. It remains open to us whether $\mathsf{PosComp}^{\mathrm{op}}$ is also a variety; however, based on the duality results of [HN16] and inspired by [Isb82], we are able to prove that $\mathsf{PosComp}^{\mathrm{op}}$ is a $\aleph_1$-ary quasivariety and give a partial description of its algebraic theory. This description turns out to be sufficient to identify also the dual of the category of coalgebras for the Vietoris functor $V\colon \mathsf{PosComp} \to \mathsf{PosComp}$ as a $\aleph_1$-ary quasivariety. Finally, we characterise the $\aleph_1$-copresentable objects of $\mathsf{PosComp}$ as precisely the metrisable ones.

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2. Preliminaries

In this section we recall the notion of partially ordered compact space introduced in [Nac50] together with some fundamental properties of these spaces.

Definition 2.1. A ***partially ordered compact space*** $(X, \leq, \tau)$ consists of a set X, a partial order $\leq$ on X and a compact topology τ on X so that

$$\{(x, y) \in X \times X \mid x \leq y\}$$

is closed in $X \times X$ with respect to the product topology.

We will often simply write X instead of $(X, \leq, \tau)$. For every partially ordered compact space X, also the subset

$$\{(x, y) \in X \times X \mid x \geq y\}$$

is closed in $X \times X$ since the mapping $X \times X \to X \times X$, $(x, y) \mapsto (y, x)$ is a homeomorphism. Therefore the diagonal

$$\Delta_X = \{(x, y) \in X \times X \mid x \leq y\} \cap \{(x, y) \in X \times X \mid x \geq y\}$$

is closed in $X \times X$, which tells us that the topology of a partially ordered compact space is Hausdorff. We denote the category of partially ordered compact spaces and monotone continuous maps by $\mathsf{PosComp}$.

Example 2.2. The unit interval $[0, 1]$ with the usual Euclidean topology and the "greater or equal" relation $\geqslant$ is a partially ordered compact space; via the mapping $x \mapsto 1 - x$, this space is isomorphic in $\mathsf{PosComp}$ to the space with the same topology and the "less or equal" relation $\leqslant$.

Clearly, there is a canonical forgetful functor $\mathsf{PosComp} \to \mathsf{Pos}$ from $\mathsf{PosComp}$ to the category Pos of partially ordered sets and monotone maps. By the observation above, forgetting the order relation defines a functor $\mathsf{PosComp} \to \mathsf{CompHaus}$ from $\mathsf{PosComp}$ to the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps. For more information regarding properties of $\mathsf{PosComp}$ we refer to [Nac65, GHK$^+$80, Jun04, Tho09]; however, we recall here:

Theorem 2.3. *The category* $\mathsf{PosComp}$ *is complete and cocomplete, and both canonical forgetful functors* $\mathsf{PosComp} \to \mathsf{CompHaus}$ *and* $\mathsf{PosComp} \to \mathsf{Pos}$ *preserve limits.*

Proof. This follows from the construction of limits and colimits in $\mathsf{PosComp}$ described in [Tho09]. □

We call an injective monotone continuous map $m\colon X \to Y$ between partially ordered compact spaces an ***embedding*** in $\mathsf{PosComp}$ whenever m is an order embedding, that is,

$$x \leq y \iff m(x) \leq m(y)$$

for all $x, y \in X$. Note that the embeddings in $\mathsf{PosComp}$ are, up to isomorphism, the closed subspace inclusions with the induced order. More generally, a cone $(f_i\colon X \to Y_i)_{i \in I}$ in $\mathsf{PosComp}$ is called initial whenever, for all $x_0, x_1 \in X$,

$$x_0 \leq x_1 \iff \forall i \in I \,.\, f_i(x_0) \leq f_i(x_1).$$

In fact, this condition is equivalent to affirm that the cone $(f_i\colon X \to Y_i)_{i \in I}$ is initial with respect to the forgetful functor $\mathsf{PosComp} \to \mathsf{CompHaus}$ (see [Tho09]). The following result of Nachbin is crucial for our work.

Theorem 2.4. *The unit interval* $[0, 1]$ *is injective in* $\mathsf{PosComp}$ *with respect to embeddings.*

Proof. See [Nac65, Theorem 6]. □

As we have shown in [HN16], the theorem above has the following important consequences.

Theorem 2.5. *The regular monomorphisms in* $\mathsf{PosComp}$ *are, up to isomorphism, the closed subspaces with the induced order. Consequently,* $\mathsf{PosComp}$ *has (Epi,Regular mono)-factorisations and the unit interval* $[0, 1]$ *is a regular injective regular cogenerator of* $\mathsf{PosComp}$.

Proof. Based on Theorem 2.4, the characterisation of regular monomorphisms as precisely the embeddings can be found in [HN16], as well as a proof for the fact that $[0,1]$ is a regular cogenerator of PosComp. □

We close this section with the following characterisation of cofiltered limits in CompHaus which goes back to [Bou66, Proposition 8, page 89] (see also [Hof02, Proposition 4.6] and [HNN16]).

Theorem 2.6. *Let $D\colon I \to \mathsf{CompHaus}$ be a cofiltered diagram and $(p_i\colon L \to D(i))_{i\in I}$ a cone for D. The following conditions are equivalent:*

(i) The cone $(p_i\colon L \to D(i))_{i\in I}$ is a limit of D.
(ii) The cone $(p_i\colon L \to D(i))_{i\in I}$ is mono and, for every $i \in I$, the image of p_i is equal to the intersection of the images of all $D(k\colon j \to i)$ with codomain i:

$$\operatorname{im} p_i = \bigcap_{j\to i} \operatorname{im} D(j \xrightarrow{k} i).$$

We emphasise that this intrinsic characterisation of cofiltered limits in CompHaus is formally dual to the following well-known description of filtered colimits in Set (see [AR94]).

Theorem 2.7. *Let $D : I \to \mathsf{Set}$ be a filtered diagram and $(c_i\colon D(i) \to C)_{i\in I}$ a compatible cocone $(c_i\colon D(i) \to C)_{i\in I}$ for D. The following conditions are equivalent:*

(i) The cocone $(c_i\colon D(i) \to C)_{i\in I}$ is a colimit of D.
(ii) The cocone $(c_i\colon D(i) \to C)_{i\in I}$ is epi and, for all $i \in I$, the coimage of c_i is equal to the cointersection of the coimages of all $D(k\colon i \to j)$ with domain i:

$$c_i(x) = c_i(y) \iff \exists (i \xrightarrow{k} j) \in I \,.\, D(k)(x) = D(k)(y),$$

and $x, y \in D(i)$.

3. The quasivariety $\mathsf{PosComp}^{\mathrm{op}}$

The principal aim of this section is to identify $\mathsf{PosComp}^{\mathrm{op}}$ as a $\aleph_1$-ary quasivariety; moreover, we give a more concrete presentation of the algebra structure of $\mathsf{PosComp}^{\mathrm{op}}$. To achieve this goal, we built on [HN16] where $\mathsf{PosComp}^{\mathrm{op}}$ is shown to be equivalent to the category of certain $[0,1]$-enriched categories, for various quantale structures on the complete lattice $[0,1]$. Arguably, the most convenient quantale structure is the ***Łukasiewicz tensor*** given by $u \odot v = \max(0, u + v - 1)$, for $u, v \in [0,1]$. For this quantale, a $[0,1]$***-category*** is a set X equipped with a mapping $a\colon X \times X \to [0,1]$ so that

$$1 \leqslant a(x,x) \qquad\text{and}\qquad a(x,y) \odot a(y,z) \leqslant a(x,z),$$

for all $x, y \in X$. Each $[0,1]$-category (X,a) induces the order relation (that is, reflexive and transitive relation)

$$x \leqslant y \text{ whenever } 1 \leqslant a(x,y) \qquad (x,y \in X)$$

on X. A $[0,1]$-category is called ***separated*** whenever this order relation is anti-symmetric. As we explain in Section 5, categories enriched in this quantale can be also thought of as metric spaces.

To state the duality result of [HN16], we need to impose certain (co)completeness conditions on $[0,1]$-categories. Since these conditions will not be used explicitly in this paper, we simply refer to [HN16] for their definitions. Eventually, we consider the category A with objects all separated finitely cocomplete $[0,1]$-categories with a monoid structure that, moreover, admit $[0,1]$-powers; the morphisms of A are the finitely cocontinuous $[0,1]$-functors preserving the monoid structure and the $[0,1]$-powers. Alternatively, these structures can be described algebraically as sup-semilattices with actions of $[0,1]$; therefore we simply refer to [Kel82, Stu14] for information about enriched categories and proceed by describing A as a category of algebras.

Remark 3.1. The category A together with its canonical forgetful functor $\mathsf{A} \to \mathsf{Set}$ is a $\aleph_1$-ary quasivariety; we recall now the presentation given in [HN16]. For more information on varieties and quasivarieties we refer to [AR94]. Firstly, the set of operation symbols consists of

- the nullary operation symbols $\bot$ and $\top$;
- the unary operation symbols $-\odot u$ and $-\pitchfork u$, for each $u\in[0,1]$;
- the binary operation symbols $\vee$ and $\circledcirc$.

Secondly, the algebras for this theory should be sup-semilattices with a supremum-preserving action of $[0,1]$; writing $x\le y$ as an abbreviation for the equation $y=x\vee y$, this translates to the equations and implications

$$x\vee x=x,\qquad x\vee(y\vee z)=(x\vee y)\vee z,\qquad x\vee\bot=x,\qquad x\vee y=y\vee x,$$
$$x\odot 1=x,\qquad (x\odot u)\odot v=x\odot(u\odot v),\qquad \bot\odot u=\bot,\qquad (x\vee y)\odot u=(x\odot u)\vee(y\odot u),$$

$$x\odot u\le x\odot v\quad\text{and}\quad \bigwedge_{u\in S}(x\odot u\le y)\implies(x\odot v\le y)\quad (S\subseteq[0,1]\text{ countable and }v=\sup S).$$

The algebras defined by the operations $\bot$, $\vee$ and $-\odot u$ ($u\in[0,1]$) and the equations above are precisely the separated $[0,1]$-categories with finite weighted colimits. Such a $[0,1]$-category (X,a) has all *powers* $x\pitchfork u$ ($x\in X, u\in[0,1]$) if and only if, for all $u\in[0,1]$, $-\odot u$ has a right adjoint $-\pitchfork u$ with respect to the underlying order. Therefore we add to our theory the implications

$$x\odot u\le y\iff x\le y\pitchfork u,$$

for all $u\in[0,1]$. Finally, regarding $\circledcirc$, we impose the commutative monoid axioms with neutral element the top-element:

$$x\circledcirc y=y\circledcirc x,\qquad x\circledcirc(y\circledcirc z)=(x\circledcirc y)\circledcirc z,\qquad x\circledcirc\top=x,\qquad \top\le x.$$

Moreover, we require this multiplication to preserve suprema and the action $-\odot u$ (for $u\in[0,1]$) in each variable:

$$x\circledcirc(y\vee z)=(x\circledcirc y)\vee(x\circledcirc z),\qquad x\circledcirc\bot=\bot,\qquad x\circledcirc(y\odot u)=(x\circledcirc y)\odot u.$$

Remark 3.2. The unit interval $[0,1]$ becomes an algebra for the theory above with $\circledcirc=\odot$ and $v\pitchfork u=\min(1,1-u+v)=1-\max(0,u-v)$, and the usual interpretation of all other symbols.

The following result is in [HN16].

Theorem 3.3. *The functor*

$$C\colon \mathsf{PosComp}^{\mathrm{op}}\longrightarrow\mathsf{A}$$

sending $f\colon X\to Y$ to $Cf\colon CY\to CX$, $\psi\mapsto\psi\cdot f$ is fully faithful, here the structure on

$$CX=\{f\colon X\to[0,1]\mid f\text{ is monotone and continuous}\}$$

is defined pointwise.

Remark 3.4. The theorem above remains valid if we augment the algebraic theory of A by further operation symbols corresponding to monotone continuous functions $[0,1]^I\to[0,1]$. More precisely, let $\aleph$ be a cardinal and $h\colon[0,1]^\aleph\to[0,1]$ be a monotone continuous map. If we add to the algebraic theory of A a operation symbol of arity $\aleph$, then $C\colon\mathsf{PosComp}^{\mathrm{op}}\to\mathsf{A}$ lifts to a fully faithful functor from $\mathsf{PosComp}^{\mathrm{op}}$ to the category of algebras for this theory by interpreting the new operation symbol in CX point-wise by h. Note that all A-morphisms of type $CY\to CX$ preserves this new operation automatically.

Remark 3.5. Note that $1-u=0\pitchfork u$, for every $u\in[0,1]$. Therefore we can express truncated minus $v\ominus u=\max(0,v-u)$ in $[0,1]$ with the operations of A:

$$v\ominus u=0\pitchfork(u\pitchfork v).$$

In particular, every subalgebra $M\subseteq CX$ of CX is also closed under truncated minus.

Since we have chosen the Łukasiewicz tensor, the categorical closure on CX (see [HT10]) coincides with the usual topology induced by the "sup-metric" on CX; in the sequel we consider this topology on $C(X)$. One important step towards the identification of the image of $C\colon \mathsf{PosComp}^{\mathrm{op}} \longrightarrow \mathsf{A}$ is the following adaption of the classical "Stone–Weierstraß theorem" (see [HN16]).

Theorem 3.6. *Let X be a partially ordered compact space and $m\colon A \hookrightarrow CX$ be a subobject of CX in A so that the cone $(m(a)\colon X \to [0,1])$ is point-separating and initial. Then m is dense in CX. In particular, if A is Cauchy complete, then m is an isomorphism.*

One important consequence of Theorem 3.6 is the following proposition.

Proposition 3.7. *The unit interval $[0,1]$ is $\aleph_1$-copresentable in* $\mathsf{PosComp}$.

Proof. This can be shown with the same argument as in [GU71, 6.5.(c)]. Firstly, by Theorem 3.6, $\hom(-,[0,1])$ sends every $\aleph_1$-codirected limit to a jointly surjective cocone. Secondly, using Theorem 2.7, this cocone is a colimit since $[0,1]$ is $\aleph_1$-copresentable in $\mathsf{CompHaus}$. ☐

Theorem 3.8. *The functor $C\colon \mathsf{PosComp}^{\mathrm{op}} \to \mathsf{A}$ corestricts to an equivalence between $\mathsf{PosComp}^{\mathrm{op}}$ and the full subcategory of A defined by those objects A which are Cauchy complete and where the cone of all A-morphisms from A to $[0,1]$ is point-separating.*

Proof. See [HN16]. ☐

Instead of working with Cauchy completeness, we wish to add an operation to the algebraic theory of A so that, if M is closed in CX under this operation, then M is closed with respect to the topology of the $[0,1]$-category CX. In the case of $\mathsf{CompHaus}$, this is achieved in [Isb82] using the operation

$$[0,1]^{\mathbb{N}} \longrightarrow [0,1],\ (u_n)_{n\in\mathbb{N}} \longmapsto \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u_n$$

on $[0,1]$; since the limit of a convergent sequence $(\varphi_n)_{n\in\mathbb{N}}$ can be calculated as

$$\lim_{n\to\infty} \varphi_n = \varphi_0 + (\varphi_1 - \varphi_0) + \dots .$$

However, this argument cannot be transported directly into the ordered setting since the difference $\varphi_1 - \varphi_0$ of two monotone maps $\varphi_0, \varphi_1\colon X \to [0,1]$ is not necessarily monotone. To circumvent this problem, we look for a monotone and continuous function $[0,1]^{\mathbb{N}} \to [0,1]$ which calculates the limit of "sufficiently many sequences". We make now the meaning of "sufficiently many" more precise.

Lemma 3.9. *Let $M \subseteq CX$ be a subalgebra in A and $\psi \in CX$ with $\psi \in \overline{M}$. Then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in M converging to ψ so that*

(1) $(\psi_n)_{n\in\mathbb{N}}$ is increasing, and
(2) for all $n \in \mathbb{N}$ and all $x \in X$: $\psi_{n+1}(x) - \psi_n(x) \leq \frac{1}{2^n}$.

Proof. We can find $(\psi_n)_{n\in\mathbb{N}}$ so that, for all $n \in \mathbb{N}$, $|\psi_n(x) - \psi(x)| \leq \frac{1}{n+1}$. Then the sequence $(\psi_n \ominus \frac{1}{n+1})_{n\in\mathbb{N}}$ converges to ψ too; moreover, since $M \subseteq CX$ is a subalgebra, also $\psi_n \ominus \frac{1}{n+1} \in M$, for all $n \in \mathbb{N}$. Therefore we can assume that we have a sequence $(\psi_n)_{n\in\mathbb{N}}$ in M with $(\psi_n)_{n\in\mathbb{N}} \to \psi$ and $\psi_n \leq \psi$, for all $n \in \mathbb{N}$. Then the sequence $(\psi_0 \vee \dots \vee \psi_n)_{n\in\mathbb{N}}$ has all its members in M, is increasing and converges to ψ. Finally, there is a subsequence of this sequence which satisfies the second condition above. ☐

Lemma 3.10. *Every increasing sequence $(u_n)_{n\in\mathbb{N}}$ in $[0,1]$ satisfying $u_{n+1} - u_n \leq \frac{1}{2^n}$, for all $n \in \mathbb{N}$, is Cauchy. Let*

$$\mathcal{C} = \{(u_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} \mid (u_n)_{n\in\mathbb{N}} \textit{ is monotone and } u_{n+1} - u_n \leq \frac{1}{2^n}, \textit{ for all } n \in \mathbb{N}\}.$$

Then every sequence in $\mathcal{C}$ is Cauchy and $\lim\colon \mathcal{C} \to [0,1]$ *is monotone and continuous.*

Motivated by the two lemmas above, we are looking for a monotone continuous map $[0,1]^{\mathbb{N}} \to [0,1]$ which sends every sequence in $\mathcal{C}$ to its limit. Such a map can be obtained by combining $\lim\colon \mathcal{C} \to [0,1]$ with a monotone continuous retraction of the inclusion map $\mathcal{C} \hookrightarrow [0,1]^{\mathbb{N}}$.

Lemma 3.11. *The map* $\mu\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}} \mapsto (u_0 \vee \dots \vee u_n)_{n\in\mathbb{N}}$ *is monotone and continuous.*

Clearly, μ sends a sequence to an increasing sequence, and $\mu((u_n)_{n\in\mathbb{N}}) = (u_n)_{n\in\mathbb{N}}$ for every increasing sequence $(u_n)_{n\in\mathbb{N}}$.

Lemma 3.12. *The map* $\gamma\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ *sending a sequence* $(u_n)_{n\in\mathbb{N}}$ *to the sequence* $(v_n)_{n\in\mathbb{N}}$ *defined recursively by*

$$v_0 = u_0 \qquad \text{and} \qquad v_{n+1} = \min\left(u_{n+1}, v_n + \frac{1}{2^n}\right)$$

is monotone and continuous. Furthermore, γ *sends an increasing sequence to an increasing sequence.*

Proof. It is easy to see that γ is monotone. In order to verify continuity, we consider $\mathbb{N}$ as a discrete topological space, this way $[0,1]^{\mathbb{N}}$ is an exponential in Top. We show that γ corresponds via the exponential law to a (necessarily continuous) map $f\colon \mathbb{N} \to [0,1]^{([0,1]^{\mathbb{N}})}$. The recursion data above translates to the conditions

$$f(0) = \pi_0 \qquad \text{and} \qquad f(n+1)((u_m)_{m\in\mathbb{N}}) = \min\left(u_{n+1}, f(n)((u_m)_{m\in\mathbb{N}}) + \frac{1}{2^n}\right),$$

that is, f is defined by the recursion data $\pi_0 \in [0,1]^{([0,1]^{\mathbb{N}})}$ and

$$[0,1]^{([0,1]^{\mathbb{N}})} \longrightarrow [0,1]^{([0,1]^{\mathbb{N}})},\ \varphi \longmapsto \min\left(\pi_{n+1}, \varphi + \frac{1}{2^n}\right).$$

Note that with $\varphi\colon [0,1]^{\mathbb{N}} \to [0,1]$ also $\min\left(\pi_{n+1}, \varphi + \frac{1}{2^n}\right) : [0,1]^{\mathbb{N}} \to [0,1]$ is continuous. Finally, if $(u_n)_{n\in\mathbb{N}}$ is increasing, then so is $(v_n)_{n\in\mathbb{N}}$. ☐

We conclude that the map $\gamma \cdot \mu\colon [0,1]^{\mathbb{N}} \to \mathcal{C}$ is a retraction for the inclusion map $\mathcal{C} \to [0,1]^{\mathbb{N}}$ in $\mathsf{PosComp}$. Therefore we define now:

Definition 3.13. Let $\overline{\mathsf{A}}$ be the $\aleph_1$-ary quasivariety obtained by adding one $\aleph_1$-ary operation symbol to the theory of A (see Remark 3.1). Then $[0,1]$ becomes an object of $\overline{\mathsf{A}}$ by interpreting this operation symbol by

$$\delta = \lim \cdot \gamma \cdot \mu\colon [0,1]^{\mathbb{N}} \to [0,1].$$

The (accordingly modified) functor $C\colon \mathsf{PosComp} \to \overline{\mathsf{A}}$ is fully faithful (see Remark 3.4); moreover, by Proposition 3.7, C sends $\aleph_1$-codirected limits to $\aleph_1$-directed colimits in $\overline{\mathsf{A}}$.

Definition 3.14. Let $\overline{\mathsf{A}}_0$ be the subcategory of $\overline{\mathsf{A}}$ defined by those objects A where the cone of all morphisms from A to $[0,1]$ is point-separating.

Hence, $\overline{\mathsf{A}}_0$ is a regular epireflective full subcategory of $\overline{\mathsf{A}}$ and therefore also a quasivariety. Moreover:

Theorem 3.15. *The embedding* $C\colon \mathsf{PosComp}^{\mathrm{op}} \to \overline{\mathsf{A}}$ *corestricts to an equivalence* $C\colon \mathsf{PosComp}^{\mathrm{op}} \to \overline{\mathsf{A}}_0$. *Hence,* $\overline{\mathsf{A}}_0$ *is closed in* $\overline{\mathsf{A}}$ *under* $\aleph_1$*-directed colimits and therefore also a* $\aleph_1$*-ary quasivariety (see* [AR94, Remark 3.32]*).*

4. Vietoris coalgebras

In this section we consider the Vietoris functor $V\colon \mathsf{PosComp} \to \mathsf{PosComp}$ and the associated category $\mathrm{CoAlg}(V)$ of coalgebras and homomorphisms. We show that $\mathrm{CoAlg}(V)$ as well as certain full subcategories are also $\aleph_1$-ary quasivarieties.

Recall from [Sch93] (see also [HN16, Proposition 3.28]) that, for a partially ordered compact space X, the elements of VX are the closed upper subsets of X, the order on VX is containment $\supseteq$, and the sets

$$\{A \in VX \mid A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ open lower}) \quad \text{and} \quad \{A \in VX \mid A \cap K = \varnothing\} \quad (K \subseteq X \text{ closed lower})$$

generate the compact Hausdorff topology on VX. Furthermore, for $f\colon X \to Y$ in $\mathsf{PosComp}$, the map $Vf\colon VX \to VY$ sends A to the up-closure $\uparrow f[A]$ of $f[A]$. A coalgebra (X, α) for V consists of a partially ordered compact space X and a monotone continuous map $\alpha\colon X \to VX$. For coalgebras (X, α) and (Y, β), a homomorphism of coalgebras $f\colon (X, \alpha) \to (Y, \beta)$ is a monotone continuous map $f\colon X \to Y$ so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ VX & \xrightarrow[Vf]{} & VY \end{array}$$

commutes. The coalgebras for the Vietoris functor and their homomorphisms form the category $\mathrm{CoAlg}(V)$, and forgetting the coalgebra structure gives rise to the canonical forgetful functor $\mathrm{CoAlg}(V) \to \mathsf{PosComp}$ that sends (X, α) to X and leaves the maps unchanged. For the general theory of coalgebras we refer to [Adá05].

As it is well-known, V is part of a monad $\mathbb{V} = (V, m, e)$ on $\mathsf{PosComp}$; here $e_X\colon X \to VX$ sends x to $\uparrow x$ and $m_X\colon VVX \to VX$ is given by $\mathcal{A} \mapsto \bigcup \mathcal{A}$. Clearly, a coalgebra structure $X \to VX$ for V can be also interpreted as an endomorphism $X \;-\!\!\!\circ\!\!\!\to X$ in the Kleisli category $\mathsf{PosComp}_{\mathbb{V}}$. In the sequel we will use this perspective together with the duality result for $\mathsf{PosComp}_{\mathbb{V}}$ of [HN16] to show that also $\mathrm{CoAlg}(V)^{\mathrm{op}}$ is a $\aleph_0$-ary quasivariety.

Let $\overline{\mathsf{B}}$ denote the category with the same objects as $\overline{A}$ and morphisms those maps $\varphi\colon A \to A'$ that preserve finite suprema and the action $- \odot u$, for all $u \in [0, 1]$, and satisfy

$$\varphi(x \circledcirc y) \le \varphi(x) \circledcirc \varphi(y),$$

for all $x, y \in A$. The functor $C\colon \mathsf{PosComp}^{\mathrm{op}} \to \overline{\mathsf{A}}$ extends to a fully faithful functor $C\colon \mathsf{PosComp}_{\mathbb{V}} \to \overline{\mathsf{B}}$ making the diagram

$$\begin{array}{ccc} \mathsf{PosComp}_{\mathbb{V}}^{\mathrm{op}} & \xrightarrow{C} & \overline{\mathsf{B}} \\ \uparrow & & \uparrow \\ \mathsf{PosComp}^{\mathrm{op}} & \xrightarrow[C]{} & \overline{\mathsf{A}}_0 \end{array}$$

commutative, where the vertical arrows denote the canonical inclusion functors. Therefore the category $\mathrm{CoAlg}(V)$ is dually equivalent to the category with objects all pairs (A, a) consisting of an $\overline{\mathsf{A}}_0$ object A and a $\overline{\mathsf{B}}$-morphism $a\colon A \to A$, and a morphism between such pairs (A, a) and (A', a') is an $\overline{\mathsf{A}}_0$-morphism $A \to A'$ commuting in the obvious sense with a and a'.

Theorem 4.1. *The category* $\mathrm{CoAlg}(V)$ *of coalgebras and homomorphisms for the Vietoris functor* $V\colon \mathsf{PosComp} \to \mathsf{PosComp}$ *is dually equivalent to a* $\aleph_1$*-ary quasivariety.*

Proof. Just consider the algebraic theory of $\overline{\mathsf{A}}_0$ augmented by one unary operation symbol and by those equations which express that the corresponding operation is a $\overline{\mathsf{B}}$-morphism. □

In particular, $\mathrm{CoAlg}(V)$ is complete and the forgetful functor $\mathrm{CoAlg}(V) \to \mathsf{PosComp}$ preserves $\aleph_1$-codirected limits. In fact, slightly more is shown in [HNN16]:

Proposition 4.2. *The forgetful functor* $\mathrm{CoAlg}(V) \to \mathsf{PosComp}$ *preserves codirected limits.*

We finish this section by pointing out some further consequences of our approach and consider certain full subcategories of $\mathrm{CoAlg}(V)$. For instance, still thinking of a coalgebra structure $\alpha\colon X \to VX$ as an endomorphism $\alpha\colon X \;-\!\!\!\circ\!\!\!\to X$ in $\mathsf{PosComp}_{\mathbb{V}}$, we say that α is ***reflexive*** whenever $1_X \le \alpha$ in $\mathsf{PosComp}_{\mathbb{V}}$, and α is called ***transitive*** whenever $\alpha \circ \alpha \le \alpha$ in $\mathsf{PosComp}_{\mathbb{V}}$. Passing now to the corresponding $\overline{B}$-morphism $a\colon A \to A$, these inequalities can be expressed as equations in A, and we conclude:

Proposition 4.3. *The full subcategory of* $\mathrm{CoAlg}(V)$ *defined by all reflexive (transitive, reflexive and transitive) coalgebras is dually equivalent to an* $\aleph_1$*-ary quasivariety. Moreover, this subcategory is coreflective in* $\mathrm{CoAlg}(V)$ *and closed under* $\aleph_1$*-ary limits.*

Proof. This follows from the discussion above and from [AR94, Theorem 1.66]. □

Another way of specifying full subcategories of $\mathrm{CoAlg}(V)$ uses coequations (see [Adá05, Definition 4.18]). More generally, for a class $\mathcal{M}$ of monomorphisms in $\mathrm{CoAlg}(V)$, a coalgebra X for V is called ***coorthogonal*** whenever, for all $m\colon A \to B$ in $\mathcal{M}$ and all homomorphisms $f\colon X \to B$ there exists a (necessarily unique) homomorphism $g\colon X \to A$ with $m \cdot g = f$ (see [AR94, Definition 1.32] for the dual notion). We write $\mathcal{M}^\top$ for the full subcategory of $\mathrm{CoAlg}(V)$ defined by those coalgebras which are coorthogonal to $\mathcal{M}$. From the dual of [AR94, Theorem 1.39] we obtain:

Proposition 4.4. *For every set* $\mathcal{M}$ *of monomorphisms in* $\mathrm{CoAlg}(V)$*, the inclusion functor* $\mathcal{M}^\top \hookrightarrow \mathrm{CoAlg}(V)$ *has a right adjoint. Moreover, if* λ *denotes a regular cardinal larger or equal to* $\aleph_1$ *so that, for every arrow* $m \in \mathcal{M}$*, the domain and codomain of* m *is* λ*-copresentable, then* $\mathcal{M}^\top \hookrightarrow \mathrm{CoAlg}(V)$ *is closed under* λ*-codirected limits.*

Corollary 4.5. *For every set of coequations in* $\mathrm{CoAlg}(V)$*, the full subcategory of* $\mathrm{CoAlg}(V)$ *defined by these coequations is coreflective.*

5. $\aleph_1$-copresentable spaces

It is shown in [GU71] that the the $\aleph_1$-copresentable objects in $\mathsf{CompHaus}$ are precisely the metrisable compact Hausdorff spaces. We end this paper with a characterisation of the $\aleph_1$-copresentable objects in $\mathsf{PosComp}$ which resembles the one for compact Hausdorff spaces; to do so, we consider generalised metric spaces in the sense of Lawvere [Law73].

More precisely, we think of metric spaces as categories enriched in the quantale $[0,1]$, ordered by the "greater or equal" relation $\geqslant$, with tensor product $\oplus$ given by truncated addition:

$$u \oplus v = \min(1, u + v),$$

for all $u, v \in [0,1]$. We note that the right adjoint $\hom(u, -)$ of $u \oplus -\colon [0,1] \to [0,1]$ is defined by

$$\hom(u, v) = v \ominus u = \max(0, v - u),$$

for all $u, v \in [0,1]$.

Remark 5.1. Via the isomorhism $[0,1] \to [0,1]$, $u \mapsto 1 - u$, the quantale described above is isomorphic to the quantale $[0,1]$ equipped with the Łukasiewicz tensor used in Section 3. However, we decided to switch so that categories enriched in $[0,1]$ look more like metric spaces.

Definition 5.2. A ***metric space*** is a pair (X, a) consisting of a set X and a map $a\colon X \times X \to [0,1]$ satisfying

$$0 \geqslant a(x,x) \qquad \text{and} \qquad a(x,y) \oplus a(y,z) \geqslant a(x,z),$$

for all $x, y, z \in X$. A map $f\colon X \to Y$ between metric spaces (X, a), (Y, b) is called ***non-expansive*** whenever

$$a(x, x') \geqslant b(f(x), f(x')),$$

for all $x, x' \in X$. Metric spaces and non-expansive maps form the category Met.

Example 5.3. The unit interval $[0,1]$ is a metric space with metric $\hom(u, v) = v \ominus u$.

Our definition of metric space is not the classical one. Firstly, we consider only metrics bounded by 1; however, since we are interested in the induced topology and the induced order, "large" distances are irrelevant. Secondly, we allow distance zero for different points, which, besides topology, also allows us to treat order theory. Every metric a on a set X defines the order relation

$$x \leq y \text{ whenever } 0 \geqslant a(x,y),$$

for all $x, y \in X$; this construction defines a functor

$$O\colon \mathsf{Met} \longrightarrow \mathsf{Ord}$$

commuting with the canonical forgetful functors to Set.

Lemma 5.4. *The functor $O\colon \mathsf{Met} \to \mathsf{Ord}$ preserves limits and initial cones.*

A metric space is called ***separated*** whenever the underlying order is anti-symmetric. Element-wise, a metric space (X, a) is separated whenever

$$(0 \geqslant a(x,y) \ \& \ 0 \geqslant a(y,z)) \implies x = y,$$

for all $x, y \in X$.

Thirdly, we are not insisting on symmetry. However, every metric space (X, a) can be symmetrised by

$$a_s(x,y) = \max(a(x,y), a(y,x)).$$

For every metric space (X, a), we consider the topology induced by the symmetrisation a_s of a. This construction defines the faithful functor

$$T\colon \mathsf{Met} \longrightarrow \mathsf{Top}.$$

We note that (X, a) is separated if and only if the underlying topology is Hausdorff. Furthermore, we recall:

Lemma 5.5. *The functor $T\colon \mathsf{Met} \to \mathsf{Top}$ preserves finite limits. In particular, T sends subspace embeddings to subspace embeddings.*

Lemma 5.6. *Let (X, a) be a separated compact metric space. Then X equipped with the order and the topology induced by the metric a becomes a partially ordered compact space.*

Proof. See [Nac65, Chapter II]. □

Example 5.7. The metric space $[0,1]$ of Example 5.3 induces the partially ordered compact Hausdorff space $[0,1]$ with the usual Euclidean topology and the "greater or equal" relation $\geqslant$.

Definition 5.8. A partially ordered compact space X is called ***metrisable*** whenever there is a metric on X which induces the order and the topology of X. We denote the full subcategory of $\mathsf{PosComp}$ defined by all metrisable spaces by $\mathsf{PosComp}_{\mathrm{met}}$.

Proposition 5.9. *$\mathsf{PosComp}_{\mathrm{met}}$ is closed under countable limits in $\mathsf{PosComp}$.*

Proof. By Lemma 5.5, $\mathsf{PosComp}_{\mathrm{met}}$is closed under finite limits in $\mathsf{PosComp}$. The argument for countable products is the same as in the classical case: for a family $(X_n)_{n\in\mathbb{N}}$ of metrisable partially ordered compact Hausdorff spaces, with the metric a_n on X_n ($n \in \mathbb{N}$), the structure of the product space $X = \prod_{n\in\mathbb{N}} X_n$ is induced by the metric a defined by

$$a((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_n(x_n, y_n),$$

for $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in X$. □

For classical metric spaces it is known that the compact spaces are subspaces of countable powers of the unit interval; this fact carries over without any trouble to our case. Before we present the argument, let us recall that, for every metric space (X, a), the cone $(a(x,-)\colon X \to [0,1])_{x\in X}$ is initial with respect to the forgetful functor $\mathsf{Met} \to \mathsf{Set}$; this is a consequence of the Yoneda Lemma for $[0,1]_{\oplus}$-categories (see [Law73]). Moreover, (X, a) is separated if and only if this cone is point-separating.

Lemma 5.10. *Let (X, a) be a compact metric space. Then there exists a countable subset $S \subseteq X$ so that the cone*

$$(a(z,-)\colon X \to [0,1])_{z\in S}$$

is initial with respect to the forgetful functor $\mathsf{Met} \to \mathsf{Set}$.

Proof. Since X is compact, for every natural number $n \geq 1$, there is a finite set S_n so that the open balls

$$\{y \in X \mid a(x,y) < \frac{1}{n} \text{ and } a(y,x) < \frac{1}{n}\}$$

with $x \in S_n$ cover X. Let $S = \bigcup_{n\geq 1} S_n$. We have to show that, for all $x, y \in X$,

$$\bigvee_{z\in S} a(z,y) \ominus a(z,x) \geqslant a(x,y).$$

To see that, let $\varepsilon = \frac{1}{n}$, for some $n \geq 1$. By construction, there is some $z \in S$ so that $a(x,z) < \varepsilon$ and $a(z,x) < \varepsilon$. Hence,

$$(a(z,y) \ominus a(z,x)) + 2\varepsilon \geqslant a(z,y) + a(x,z) \geqslant a(x,y);$$

and the assertion follows. ☐

Proposition 5.11. *Every partially ordered compact space is a $\aleph_1$-cofiltered limit in* $\mathsf{PosComp}$ *of metrisable spaces.*

Proof. For a separated metric space $X = (X, a)$, the initial cone $(a(x,-)\colon X \to [0,1])_{x\in S}$ of Lemma 5.10 is automatically point-separating, therefore there is an embedding $X \hookrightarrow [0,1]^{\mathbb{N}}$ in Met. This proofs that the full subcategory $\mathsf{PosComp}_{\mathrm{met}}$ of $\mathsf{PosComp}$ is small. Let X be a partially ordered compact space. By Proposition 5.9, the canonical diagram

$$D\colon X \downarrow \mathsf{PosComp}_{\mathrm{met}} \longrightarrow \mathsf{PosComp}$$

is $\aleph_1$-cofiltered. Moreover, the canonical cone

$$(f\colon X \to Y)_{f\in(X\downarrow \mathsf{PosComp}_{\mathrm{met}})} \tag{5.i}$$

is initial since (5.i) includes the cone $(f\colon X \to [0,1])_f$. Finally, to see that (5.i) is a limit cone, we use Theorem 2.6: for every $f\colon X \to Y$ with Y metrisable, $\operatorname{im} f \hookrightarrow Y$ actually belongs to $\mathsf{PosComp}_{\mathrm{met}}$, which proves

$$\operatorname{im} f = \bigcap_{k\colon g\to f\in(X\downarrow \mathsf{PosComp}_{\mathrm{met}})} \operatorname{im} D(k).$$ ☐

Corollary 5.12. *Every $\aleph_1$-copresentable object in* $\mathsf{PosComp}$ *is metrisable.*

Proof. Also here the argument is the same as for $\mathsf{CompHaus}$. Let X be a $\aleph_1$-copresentable object in $\mathsf{PosComp}$. By Proposition 5.11, we can present X as a limit $(p_i\colon X \to X_i)_{i\in I}$ of a $\aleph_1$-codirected diagram $D\colon I \to \mathsf{PosComp}$ where all $D(i)$ are metrisable. Since X is $\aleph_1$-copresentable, the identity $1_X\colon X \to X$ factorises as

$$X \xrightarrow{p_i} X_i \xrightarrow{h} X,$$

for some $i \in I$. Hence, being a subspace of a metrisable space, X is metrisable. ☐

To prove that every metrisable partially ordered compact space X is $\aleph_1$-copresentable, we will show that every closed subspace $A \hookrightarrow [0,1]^I$ with I countable is an equaliser of a pair of arrows

$$[0,1]^I \rightrightarrows [0,1]^J$$

with also J being countable. For a symmetric metric on X, one can simply consider

$$[0,1]^I \underset{0}{\overset{a(A,-)}{\rightrightarrows}} [0,1],$$

but in our non-symmetric setting this argument does not work. We start with an auxiliary result which follows from Theorem 2.4. Our argument here is a slight modification of the one used in the characterisation of regular monomorphisms in $\mathsf{PosComp}$ obtained in [HN16].

Lemma 5.13. *Let X be a partially ordered compact space, $A, B \subseteq X$ closed subsets so that $A \cap B = \varnothing$ and $B = {\downarrow}B \cap {\uparrow}B$. Then there is a family $(f_u \colon X \to [0,1])_{u \in [0,1]}$ of monotone continuous maps which all coincide on A and, moreover, satisfy $f_u(y) = u$, for all $u \in [0,1]$ and $y \in B$.*

Proof. Put $A_0 = A \cap {\uparrow}B$ and $A_1 = A \cap {\downarrow}B$. Then A_0 and A_1 are closed subsets of X and

$$A_0 \cap A_1 = A \cap {\uparrow}B \cap {\downarrow}B = A \cap B = \varnothing.$$

Moreover, for every $x_0 \in A_0$ and $x_1 \in A_1$, $x_0 \not\leq x_1$. In fact, if $x_0 \geq y_0 \in B$ and $x_1 \leq y_1 \in B$, then $x_0 \leq x_1$ implies

$$y_0 \leq x_0 \leq x_1 \leq y_1,$$

hence $x_0 \in B$ which contradicts $A \cap B = \varnothing$. We define now the monotone continuous map

$$\begin{aligned} g \colon A_0 \cup A_1 &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} 0 & \text{if } x \in A_0, \\ 1 & \text{if } x \in A_1. \end{cases} \end{aligned}$$

By [Nac65, Theorem 6], g extends to a monotone continuous map $g \colon A \to [0,1]$. Let now $u \in [0,1]$. We define

$$\begin{aligned} f_u \colon A \cup B &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} g(x) & \text{if } x \in A, \\ u & \text{if } x \in B. \end{cases} \end{aligned}$$

Using again [Nac65, Theorem 6], f_u extends to a monotone continuous map $f_u \colon X \to [0,1]$. ☐

Corollary 5.14. *Let $n \in \mathbb{N}$ and $A \subseteq [0,1]^n$ be a closed subset. Then there exist countable set J and monotone continuous maps*

$$[0,1]^n \underset{k}{\overset{h}{\rightrightarrows}} [0,1]^J$$

so that $A \hookrightarrow [0,1]^n$ is the equaliser of h and k. In particular, A is $\aleph_1$-copresentable.

Proof. We denote by d the usual Euclidean metric on $[0,1]^n$. For every $x \in [0,1]^n$ with $x \notin A$, there is some $\varepsilon > 0$ so that the closed ball

$$B(x,\varepsilon) = \{y \in [0,1]^n \mid d(x,y) \leqslant \varepsilon\}$$

does not intersect A. Furthermore, $B = {\uparrow}B \cap {\downarrow}B$. Put

$$J = \{(k, x_1, \dots x_n) \mid k \in \mathbb{N}, k \geq 1 \text{ and } x = (x_1, \dots x_n) \in ([0,1] \cap \mathbb{Q})^n \text{ and } B(x, \frac{1}{k}) \cap A = \varnothing\};$$

clearly, J is countable. For every $j = (k, x_1, \dots x_n) \in J$, we consider the monotone continuous maps $f_0, f_1 \colon [0,1]^n \to [0,1]$ obtained in Lemma 5.13 and put $h_j = f_0$ and $k_j = f_1$. Then $A \hookrightarrow [0,1]^n$ is the equaliser of

$$[0,1]^n \underset{k = \langle k_j \rangle}{\overset{h = \langle h_j \rangle}{\rightrightarrows}} [0,1]^J$$ ☐

Theorem 5.15. *Every metrisable partially ordered compact space is $\aleph_1$-copresentable in* $\mathsf{PosComp}$.

Proof. Let X be a metrisable partially ordered compact space. By Lemma 5.10, there is an embedding $m \colon X \hookrightarrow [0,1]^{\mathbb{N}}$ in $\mathsf{PosComp}$. Moreover, with

$$J = \{F \subseteq \mathbb{N} \mid F \text{ is finite}\},$$

$(\pi_F \colon [0,1]^{\mathbb{N}} \to [0,1]^F)_{F \in J}$ is a limit cone of the codirected diagram

$$J^{\mathrm{op}} \longrightarrow \mathsf{PosComp}$$

sending F to $[0,1]^F$ and $G \supseteq F$ to the canonical projection $\pi\colon [0,1]^G \to [0,1]^F$. For every $F \in J$, we consider the (Epi,Regular mono)-factorisation

$$X \xrightarrow{p_F} X_F \xrightarrow{m_F} [0,1]^F$$

of $\pi_F \cdot m\colon X \to [0,1]^F$. Then, using again Bourbaki's criterion (see Theorem 2.6),

$$(p_F\colon X \to A_F)_{F\in J}$$

is a limit cone of the codirect diagram

$$J^{\mathrm{op}} \longrightarrow \mathsf{PosComp}$$

sending F to X_F and $G \supseteq F$ to the diagonal of the factorisation. By Corollary 5.14, each X_F is $\aleph_1$-copresentable, hence also X is $\aleph_1$-copresentable since X is a countable limit of $\aleph_1$-copresentable objects. □

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Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `dirk@ua.pt`

INESC TEC (HASLab) & Universidade do Minho, Portugal
E-mail address: `nevrenato@di.uminho.pt`

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `a28224@ua.pt`