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DYNAMICS OF HUMAN DECISIONS

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Abstract. We study a dichotomous decision model, where individuals can make the decision yes or no and can influence the decisions of others. We characterize all decisions that form Nash equilibria. Taking into account the way individuals influence the decisions of others, we construct the decision tilings where the axes reflect the personal preferences of the individuals for making the decision yes or no. These tilings characterize geometrically all the pure and mixed Nash equilibria. We show, in these tilings, that Nash equilibria form degenerated hystereses with respect to the replicator dynamics, with the property that the pure Nash equilibria are asymptotically stable and the strict mixed equilibria are unstable. These hystereses can help to explain the sudden appearance of social, political and economic crises. We observe the existence of limit cycles for the replicator dynamics associated to situations where the individuals keep changing their decisions along time, but exhibiting a periodic repetition in their decisions. We introduce the notion of altruist and individualist leaders and study the way that the leader can affect the individuals to make the decision that the leader pretends.

1. **Introduction.** The main goal in Planned Behavior or Reasoned Action theories, as developed in the works of Ajzen (see [1]) and Baker (see [4]), is to understand and predict the way individuals turn intentions into behaviors. Almeida-Cruz-Ferreira-Pinto (see [2, 9]) developed a game theoretical model for reasoned action, inspired by the works of J. Cownley and M. Wooders (see [7]). Here, we study the Pinto's dichotomous decision model (see [9, 11]), which is a simplified version of the Almeida-Cruz-Ferreira-Pinto decision model. In this model, there are just two types

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$t \in \{t_1, t_2\}$ of individuals and two possible decisions d that individuals can make. In this case, they have to choose between yes or no, i.e. $d \in \{Y, N\}$. The yes-no decision model incorporates, in the coordinates of the preference decision matrix how much an individual with type t_1 or with type t_2 likes or dislikes, to make a decision $d \in \{Y, N\}$. The yes-no decision model incorporates, in the coordinates of the preference neighbors and no-neighbors matrices, the preference that individuals with a certain type t_i have for other individuals, with the same or a different type t_j, to make the same decision or the opposite decision as theirs (see [5, 10]). The preference decision matrix and the neighbors and no-neighbors matrices can be very complex to find explicitly in real cases because they encode, for instance, information from economic, educational, political, psychological and social variables. However, if we have a qualitative or rough knowledge of these matrices, we can obtain relevant information on how individuals make decisions and why to make decisions can be so complex.

We characterize all the pure Nash equilibria and we show that the pure Nash equilibria are, in general, asymptotically stable with respect to the replicator dynamics. The pure Nash equilibria are either cohesive, i.e. all individuals with the same preferences make the same decision, or disparate, i.e. there are individuals with the same preferences that make opposite decisions. The disparate pure Nash equilibria can correspond to conflicting decisions that divide a community. We characterize all the strict mixed Nash equilibria and we prove that the strict mixed Nash equilibria are, in general, unstable. Fixing the parameters of the preference neighbors matrices we construct tilings in the plane, where the horizontal axis represent the relative preference of individuals with type t_1 to make the decision yes or no, the vertical axis represent the relative preference of individuals with type t_2 to make the decision yes or no and the pure and mixed Nash equilibria form the tiles. We prove that the tilings give a full geometrically characterization of the pure and mixed Nash equilibria.

We say that individuals with a certain type t_j have a positive influence over individuals with the same or other type t_i if the individuals with type t_i prefer to make the same decision as the individuals with type t_j and we say that individuals with type t_j have a negative influence over individuals with type t_i if the individuals with type t_i prefer to make the opposite decision from the individuals with type t_j. If all the individuals have a positive influence over individuals with the same type then there are no disparate Nash equilibria. However, if there are individuals with a certain type t_j that have a negative influence over individuals with the same type t_j then there are disparate Nash equilibria that are asymptotically stable.

The stable manifolds of the strict mixed Nash equilibria can be locally characterized by appropriate symmetries of the model. They are the main reason for certain decision strategies to persist for long periods of time before breaking down and converge to quite different strategies of decision. They also explain, partially, the complexity of the non-intuitive successive reversal of the individuals decisions along time before converging to a stable equilibrium, i.e. a high number of individuals have to keep modifying their decisions through the transient dynamics before reaching the equilibrium. Furthermore, we observe the existence of stable periodic orbits for the replicator dynamics, i.e. the individuals decisions keep changing along time exhibiting a periodic pattern. The replicator dynamics equilibria form hystereses that provide insight into how small changes in economic, educational, political, psychological or social variables can reverse abruptly individual and collective decisions. These changes in the collective decisions can lead to serious political, social or economic transformations in society.

Following [3], we introduce the notion of altruist and individualist leaders. The altruist and individualist leaders offer, respectively, advantages or disadvantages to the individuals of both type. The leader is biased if the leader offers advantages to individuals with a certain type and disadvantages to individuals with the other type. The individuals can increase or decrease the advantages or disadvantages offered by the leader depending upon their own abilities characterized by their type. We study the way that the leader can affect the individuals (potential followers) to make the decision that the leader pretends.

2. **Yes-No Decision Model.** As in [11], the *yes-no decision model* has two types $\mathbf{T} = \{t_1, t_2\}$ of individuals. Let $I_1 = \{1, \ldots, n_1\}$ be the set of all individuals with type t_1, and let $I_2 = \{1, \ldots, n_2\}$ be the set of all individuals with type t_2. Let $I = I_1 \sqcup I_2$. The individual $i \in \mathbf{I}$ has to make one decision $d \in \mathbf{D} = \{Y, N\}$[1]. Let $\mathcal{L}$ be the *preference decision matrix* whose *coordinates* γ_p^d indicate how much an individual with type t_p likes or dislikes, to make decision d

$$\mathcal{L} = \begin{pmatrix} \gamma_1^Y & \gamma_1^N \\ \gamma_2^Y & \gamma_2^N \end{pmatrix}.$$

The preference decision matrix indicates for each type of individuals the decision that the individuals prefer, i.e. the taste type of the individuals (see [2, 6, 7, 9, 11]). Let $\mathcal{N}_d$ be the *preference neighbors matrix* whose *coordinates* β_{pq}^d indicate how much an individual with type t_p who decides d likes or dislikes that an individual with type t_q also makes decision d

$$\mathcal{N}_d = \begin{pmatrix} \beta_{11}^d & \beta_{12}^d \\ \beta_{21}^d & \beta_{22}^d \end{pmatrix}.$$

Let $\overline{\mathcal{N}}_d$ be the *preference non-neighbors matrix* whose *coordinates* $\overline{\beta}_{pq}^d$ indicate how much an individual with type t_p who decide d, likes or dislikes that an individual with type t_q makes decision $d' \neq d$

$$\overline{\mathcal{N}}_d = \begin{pmatrix} \overline{\beta}_{11}^d & \overline{\beta}_{12}^d \\ \overline{\beta}_{21}^d & \overline{\beta}_{22}^d \end{pmatrix}.$$

The preference neighbors and non-neighbors matrices indicate, for each type of individuals whose decision is d, whom they prefer, or do not prefer, to be with in each decision, i.e. the crowding type of the individuals (see [2, 6, 7, 9]).

We describe the (pure) decision of the individuals by a *(pure) strategy map* $S : \mathbf{I} \to \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ its decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$ be the space of all strategies S. Given a strategy S, let $\mathcal{O}_S$ be the *strategic decision matrix* whose coordinates $l_p^d = l_p^d(S)$ indicate the number of individuals with type t_p, who make decision d

$$\mathcal{O}_S = \begin{pmatrix} l_1^Y & l_1^N \\ l_2^Y & l_2^N \end{pmatrix}.$$

The *strategic decision vector* associated to a strategy S is the vector $(l_1, l_2) = (l_1^y(S), l_2^y(S))$. Hence, l_1 (resp. $n_1 - l_1$) is the number of individuals with type t_1 who make the decision Y (resp. N). Similarly, l_2 (resp. $n_2 - l_2$) is the number

[1]Similarly, we can consider that there is a single individual with type t_p that has to make n_p decisions, or we can, also, consider a mixed model using these two possibilities.

of individuals with type t_2 who make the decision Y (resp. N). The set $\mathbf{O}$ of all possible *strategic decision vectors* is

$$\mathbf{O} = \{0, \ldots, n_1\} \times \{0, \ldots, n_2\} .$$

Let:

- $\omega_1^Y = \gamma_1^Y + \overline{\beta}_{11}^Y (n_1 - 1) + \overline{\beta}_{12}^Y n_2$;
- $\omega_1^N = \gamma_1^N + \overline{\beta}_{11}^N (n_1 - 1) + \overline{\beta}_{12}^N n_2$;
- $\omega_2^Y = \gamma_2^Y + \overline{\beta}_{22}^Y (n_2 - 1) + \overline{\beta}_{21}^Y n_1$;
- $\omega_2^N = \gamma_2^N + \overline{\beta}_{22}^N (n_2 - 1) + \overline{\beta}_{21}^N n_1$;
- $\alpha_{ij}^d = \beta_{ij}^d - \overline{\beta}_{ij}^d$, for $i, j \in \{1, 2\}$ and $d \in \{Y, N\}$.

Let $U_1 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ the *utility function* of an individual with type t_1 be given by

$$\begin{aligned} U_1(Y; l_1, l_2) &= \gamma_1^Y + \beta_{11}^Y (l_1 - 1) + \beta_{12}^Y l_2 + \overline{\beta}_{11}^Y (n_1 - l_1) + \overline{\beta}_{12}^Y (n_2 - l_2) \\ &= \omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2; \\ U_1(N; l_1, l_2) &= \gamma_1^N + \beta_{11}^N (n_1 - l_1 - 1) + \beta_{12}^N (n_2 - l_2) + \overline{\beta}_{11}^N l_1 + \overline{\beta}_{12}^N l_2 \\ &= \omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2). \end{aligned}$$

Let $U_2 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ the *utility function* of an individual with type t_2 be given by

$$\begin{aligned} U_2(Y; l_1, l_2) &= \gamma_2^Y + \beta_{22}^Y (l_2 - 1) + \beta_{21}^Y l_1 + \overline{\beta}_{22}^Y (n_2 - l_2) + \overline{\beta}_{21}^Y (n_1 - l_1) \\ &= \omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1; \\ U_2(N; l_1, l_2) &= \gamma_2^N + \beta_{22}^N (n_2 - l_2 - 1) + \beta_{21}^N (n_1 - l_1) + \overline{\beta}_{22}^N l_2 + \overline{\beta}_{21}^N l_1 \\ &= \omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1). \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$ of an individual i with type $t_{p(i)}$ is given by $U_{p(i)}(S(i); l_1^y(S), l_2^y(S))$.

Definition 2.1. Let $x = \omega_1^Y - \omega_1^N$ be the *horizontal relative decision preference* of the individuals with type t_1 and let $y = \omega_2^Y - \omega_2^N$ be the *vertical relative decision preference* of the individuals with type t_2. Let $A_{ij} = \alpha_{ij}^Y + \alpha_{ij}^N$, for $i, j \in \{1, 2\}$, be the coordinates of the *influence matrix*.

If $x > 0$, the individuals with type t_1 prefer to decide Y, without taking into account the influence of the others. If $x = 0$, the individuals with type t_1 are indifferent to decide Y or N, without taking into account the influence of the others. If $x < 0$, the individuals with type t_1 prefer to decide N, without taking into account the influence of the others.

If $A_{ij} > 0$, the individuals with type t_j have a *positive influence* over the utility of the individuals with type t_i. If $A_{ij} = 0$, the individuals with type t_j are *indifferent* for the utility of the individuals with type t_i. If $A_{ij} < 0$, the individuals with type t_j have a *negative influence* over the utility of the individuals with type t_i.

3. **Cohesive Nash equilibria.** We will show that the relative decision preferences and the influence matrix together with the total number of individuals of each type encode all the relevant information for characterizing the Nash equilibria.

A strategy $S^* : \mathbf{I} \to \mathbf{D}$ is a *(pure) Nash equilibrium* if

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$, with the property that $S^*(j) = S(j)$ for every individual $j \in I \setminus \{i\}$. The *Nash domain* $\mathbf{N}(S)$ of a strategy $S \in \mathbf{S}$ is the set of all pairs (x, y) for which S is a Nash equilibrium.

Definition 3.1. A *cohesive strategy*[2] is a strategy in which all individuals with the same type prefer to make the same decision. A *disparate strategy* is a pure strategy that is not cohesive.

As in [11], we construct the Nash domains $\mathbf{N}(S)$ for the cohesive strategies. We observe that there are four cohesive strategies: (Y, Y) *strategy*: all individuals make the decision Y; (Y, N) *strategy*: all individuals, with type t_1, make the decision Y, and all individuals, with type t_2, make the decision N; (N, Y) *strategy*: all individuals, with type t_1, make the decision N and all individuals, with type t_2, make the decision Y; (N, N) *strategy*: all individuals make the decision N.

The *horizontal* $H(Y, Y)$ and *vertical* $V(Y, Y)$ *strategic thresholds* of the (Y, Y) strategy are

$$H(Y, Y) = -\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2 \quad \text{and} \quad V(Y, Y) = -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1 \ .$$

The *Nash domain* $\mathbf{N}(Y, Y)$ is the right-upper quadrant

$$\mathbf{N}(Y, Y) = \{(x, y) : x \geq H(Y, Y) \quad \text{and} \quad y \geq V(Y, Y)\} \ .$$

The *horizontal* $H(Y, N)$ and *vertical* $V(Y, N)$ *strategic thresholds* of the (Y, N) strategy are

$$H(Y, N) = -\alpha_{11}^{Y}(n_1 - 1) + \alpha_{12}^{N} n_2 \quad \text{and} \quad V(Y, N) = \alpha_{22}^{N}(n_2 - 1) - \alpha_{21}^{Y} n_1 \ .$$

The *Nash domain* $\mathbf{N}(Y, N)$ is the right-lower quadrant

$$\mathbf{N}(Y, N) = \{(x, y) : x \geq H(Y, N) \quad \text{and} \quad y \leq V(Y, N)\}.$$

The *horizontal* $H(N, Y)$ and *vertical* $V(N, Y)$ *strategic thresholds* of the (N, Y) strategy are

$$H(N, Y) = \alpha_{11}^{N}(n_1 - 1) - \alpha_{12}^{Y} n_2 \quad \text{and} \quad V(N, Y) = -\alpha_{22}^{Y}(n_2 - 1) + \alpha_{21}^{N} n_1 \ .$$

The *Nash domain* $\mathbf{N}(N, Y)$ is the left-upper quadrant

$$\mathbf{N}(N, Y) = \{(x, y) : x \leq H(N, Y) \quad \text{and} \quad y \geq V(N, Y)\}.$$

The *horizontal* $H(N, N)$ and *vertical* $V(N, N)$ strategic thresholds of the (N, N) strategy are

$$H(N, N) = \alpha_{11}^{N}(n_1 - 1) + \alpha_{12}^{N} n_2 \quad \text{and} \quad V(N, N) = \alpha_{22}^{N}(n_2 - 1) + \alpha_{21}^{N} n_1 \ .$$

The *Nash domain* $\mathbf{N}(N, N)$ is the left-lower quadrant

$$\mathbf{N}(N, N) = \{(x, y) : x \leq H(N, N) \quad \text{and} \quad y \leq V(N, N)\} \ .$$

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[2] or equivalently, *no-split strategy* or *heard strategy*

4. **Disparate Nash equilibria.** An (l_1, l_2) *strategic set* is the set of all pure strategies $S \in \mathbf{S}$ with $l_1(S) = l_1$ and $l_2(S) = l_2$. An (l_1, l_2) *cohesive strategic set* is an (l_1, l_2) strategy set with $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. An (l_1, l_2) *disparate strategic set* is an (l_1, l_2) strategic set that is not cohesive. We observe that a cohesive strategic set has a single strategy and a disparate strategic set has more than one strategy.

Since individuals with the same type are identical, a strategy to be a Nash equilibrium depends only upon the number of individuals of each type that decide either Y or N, and not upon the individual who is making the decision.

Definition 4.1. An (l_1, l_2) *pure Nash equilibrium (set)* is an (l_1, l_2) strategic set whose strategies are Nash equilibria. The *(pure) Nash domain* $\mathbf{N}(l_1, l_2)$ is the set of all pairs (x, y) for which the (l_1, l_2) strategic set is a Nash equilibrium set.

The (l_1, l_2) pure Nash equilibrium set is *cohesive* if $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. The (l_1, l_2) pure Nash equilibrium set is *disparate* if $l_1 \notin \{0, n_1\}$ or $l_2 \notin \{0, n_2\}$.

Lemma 4.2. *Let (l_1, l_2) be a Nash equilibrium.*

(i): *If $A_{11} > 0$, then $l_1 \in \{0, n_1\}$.*
(ii): *If $A_{22} > 0$, then $l_2 \in \{0, n_2\}$.*

Furthermore, if $A_{11} > 0$ and $A_{22} > 0$, then (l_1, l_2) is cohesive.

Hence, if the individuals with a given type have a positive influence over the utility of the individuals with the same type, i.e. $A_{11} > 0$ and $A_{22} > 0$, then there are no disparate Nash equilibria.

Proof. Suppose, by contradiction, that the (l_1, l_2) strategy is a Nash equilibrium for $l_1 \in \{1, \ldots, n_1 - 1\}$. Hence, the following two inequalities hold

$$U_1(Y; l_1, l_2) \geq U_1(N; l_1 - 1, l_2) \ , \ U_1(N; l_1, l_2) \geq U_1(Y; l_1 + 1, l_2) \ .$$

By rearranging the terms in the previous inequalities, we obtain $A_{11} \leq 0$ which contradicts that A_{11} is positive. Hence, Lemma 4.2 (i) holds. The proof of the other cases follow similarly to the proof of the first case. □

The *cohesive horizontal vector* $\vec{H}$ is

$$\vec{H} = (H(Y, N) - H(N, N), V(Y, N) - V(N, N)) = -(A_{11}(n_1 - 1), A_{21} n_1) \ .$$

The *cohesive vertical vector* $\vec{V}$ is

$$\vec{V} = (H(N, Y) - H(N, N), V(N, Y) - V(N, N)) = -(A_{12} n_2, A_{22}(n_2 - 1)) \ .$$

The *disparate vector* $\vec{Z}(l_1, l_2)$ is

$$\begin{aligned} \vec{Z}(l_1, l_2) &= -l_1(A_{11}, A_{21}) - l_2(A_{12}, A_{22}) = \frac{l_1}{n_1}(\vec{H} - (A_{11}, 0)) + \frac{l_2}{n_2}(\vec{V} - (0, A_{22})) \\ &= \frac{l_1}{n_1 - 1}(\vec{H} + (0, A_{21})) + \frac{l_2}{n_2 - 1}(\vec{V} + (0, A_{22})). \end{aligned} \tag{1}$$

Lemma 4.3. *Let $l_1 \in \{1, 2, \ldots, n_1 - 1\}$ and $l_2 \in \{1, 2, \ldots, n_2 - 1\}$.*

(i): *If $A_{11} \leq 0$, the disparate Nash domain $\boldsymbol{N}(l_1, 0)$ is given by*

$$\{(H(N,N), V(N,N)) + \vec{Z}(l_1, 0) + (pA_{11}, qA_{22}) : p \in [0,1],\ q \in [0, +\infty)\}$$

and the disparate Nash domain $\boldsymbol{N}(l_1, n_2)$ is given by

$$\{(H(N,N), V(N,N)) + \vec{Z}(l_1, n_2) + (pA_{11}, qA_{22}) : p \in [0,1],\ q \in (-\infty, 1]\}\ .$$

(ii): *If $A_{22} \leq 0$, the disparate Nash domain $\boldsymbol{N}(0, l_2)$ is given by*

$$\{(H(N,N), V(N,N)) + \vec{Z}(0, l_2) + (pA_{11}, qA_{22}) : p \in [0, +\infty),\ q \in [0,1]\}$$

and the disparate Nash domain $\boldsymbol{N}(n_1, l_2)$ is given by

$$\{(H(N,N), V(N,N)) + \vec{Z}(n_1, l_2) + (pA_{11}, qA_{22}) : p \in (-\infty, 1],\ q \in [0,1]\}\ .$$

(iii): *If $A_{11} \leq 0$ and $A_{22} \leq 0$, the disparate Nash domain $\boldsymbol{N}(l_1, l_2)$ is*

$$\boldsymbol{N}(l_1, l_2) = \{(H(N,N), V(N,N)) + \vec{Z}(l_1, l_2) + (pA_{11}, qA_{22}) : p, q \in [0,1]\}\ .$$

Hence, if the individuals with a given type have a non-positive influence over the utility of the individuals with the same type, i.e. $A_{11} \leq 0$ and $A_{22} \leq 0$, then for every (l_1, l_2) disparate strategic set there are relative preferences for which (l_1, l_2) is a Nash equilibrium set.

Proof. The $(l_1, 0)$ strategy is a Nash equilibrium if, and only if, the following three inequalities hold

$$U_1(Y; l_1, 0) \geq U_1(N; l_1 - 1, 0)\ ,\ U_1(N; l_1, 0) \geq U_1(Y; l_1 + 1, 0)\ ,$$

and

$$U_2(N; l_1, 0) \geq U_2(Y; l_1, 1)\ .$$

Hence, the proof of Lemma 4.3 (i) follows from rearranging the terms in the previous inequalities. The proof of Lemma 4.3 (ii) follows similarly to the proof of the first case. Let us prove Lemma 4.3 (iii). The (l_1, l_2) strategy is a Nash equilibrium if, and only if, the following four inequalities hold

$$U_1(Y; l_1, l_2) \geq U_1(N; l_1 - 1, l_2)\ ,\ U_1(N; l_1, l_2) \geq U_1(Y; l_1 + 1, l_2)$$

and

$$U_2(Y; l_1, l_2) \geq U_2(N; l_1, l_2 - 1)\ ,\ U_2(N; l_1, l_2) \geq U_2(Y; l_1, l_2 + 1)\ .$$

Hence, the proof of Lemma 4.3 (iii) follows from rearranging the terms in the previous inequalities. □

5. **Mixed Nash equilibria.** Recall that $I = I_1 \sqcup I_2$. We describe the (mixed) decision of the individuals by a *(mixed) strategy map* $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ that associates to each individual $i \in \mathbf{I_1}$ the probability $p_i = S(i)$ to decide $Y \in \mathbf{D}$ and to each individual $j \in \mathbf{I_2}$ the probability $q_j = S(j)$ to decide $Y \in \mathbf{D}$. Hence, each individual $i \in \mathbf{I_1}$ decides $N \in \mathbf{D}$ with probability $1 - p_i = 1 - S(i)$ and each individual $j \in \mathbf{I_2}$ decides $N \in \mathbf{D}$ with probability $1 - q_j = 1 - S(j)$. We assume that the decisions of the individuals are independent.

Define $P = \sum_{i=1}^{n_1} p_i$, $Q = \sum_{j=1}^{n_2} q_j$, $P_i = P - p_i$ and $Q_j = Q - q_j$. For every individual $i \in \mathbf{I_1}$, the *Y-fitness function* $f_{Y,1} : [0,1] \times [0, n_1] \times [0, n_2] \to \mathbb{R}^+$ is given by

$$f_{Y,1}(p_i; P, Q) = \omega_1^Y + \alpha_{11}^Y P_i + \alpha_{12}^Y Q\ ;$$

and the *N-fitness function* $f_{N,1} : [0,1] \times [0, n_1] \times [0, n_2] \to \mathbb{R}^+$ is given by

$$f_{N,1}(p_i; P, Q) = \omega_1^N + \alpha_{11}^N (n_1 - 1 - P_i) + \alpha_{12}^N (n_2 - Q)\ .$$

For every individual $j \in \mathbf{I_2}$, the *Y-fitness function* $f_{Y,2} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{Y,2}(q_j;P,Q) = \omega_2^Y + \alpha_{22}^Y Q_j + \alpha_{21}^Y P \ ;$$

and the *N-fitness function* $f_{N,2} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{N,2}(q_j;P,Q) = \omega_2^N + \alpha_{22}^N (n_2 - 1 - Q_j) + \alpha_{21}^N (n_1 - P) \ .$$

Lemma 5.1. *Let $S : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$ be a mixed strategy. For every individual $i \in \mathbf{I_1}$, the utility function $U_1 : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by*

$$U_1(p_i;P,Q) = p_i \ f_{Y,1}(p_i;P,Q) + (1-p_i) \ f_{N,1}(p_i;P,Q) \ .$$

For every individual $j \in \mathbf{I_2}$, the utility function $U_2 : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$U_2(q_j,P,Q) = q_j \ f_{Y,2}(q_j;P,Q) + (1-q_j) \ f_{N,2}(q_j;P,Q) \ .$$

Proof. The proof follows by induction on the number of individuals. Let $(n_1,n_2) = (1,1)$. The utility function of individual $i=1$, with type t_1, is given by

$$\begin{aligned} U_1(p_i;p_i,q_j) &= p_i \Big(q_j f_{Y,1}(1;1,1) + (1-q_j) f_{Y,1}(1;1,0) \Big) \\ &\quad + (1-p_i) \Big(q_j f_{N,1}(0;0,1) + (1-q_j) f_{N,1}(0;0,0) \Big) \ . \end{aligned}$$

By substituting the fitness functions in the previous identity, we obtain

$$\begin{aligned} U_1(p_i;p_i,q_j) = p_i \Big(q_j \Big(\omega_1^Y + \alpha_{11}^Y 0 + \alpha_{12}^Y 1 \Big) + (1-q_j) \Big(\omega_1^Y + \alpha_{11}^Y 0 + \alpha_{12}^Y 0 \Big) \Big) \\ + (1-p_i) \Big(q_j \Big(\omega_1^N + \alpha_{11}^N 0 + \alpha_{12}^N 0 \Big) + (1-q_j) \Big(\omega_1^N + \alpha_{11}^N 0 + \alpha_{12}^N 1 \Big) \Big) \ . \end{aligned}$$

After rearranging the terms, the last identity becomes

$$\begin{aligned} U_1(p_i;p_i,q_j) &= p_i \Big(\omega_1^Y + \alpha_{12}^Y q_j \Big) + (1-p_i) \Big(\omega_1^N + \alpha_{12}^N (1-q_j) \Big) \\ &= p_i f_{Y,1}(p_i;p_i,q_j) + (1-p_i) f_{N,1}(p_i;p_i,q_j) \ . \end{aligned}$$

Similarly, the utility function of individual $j=1$, with type t_2, is given by

$$U_2(q_j;p_i,q_j) = q_j f_{Y,2}(q_j;p_i,q_j) + (1-q_j) f_{N,2}(q_j;p_i,q_j) \ .$$

Let us add one more individual $i = n_1 + 1$, with type t_1, and compute its utility function. Let us suppose, by induction, that the utility functions are known for n_1 individuals with type t_1 and for n_2 individuals with type t_2. Let $\underline{P} = \sum_{k=1}^{n_1} p_k$, $Q = \sum_{k=1}^{n_2} q_k$ and $P = \underline{P} + p_{n_1+1}$. The utility function of the individual n_1+1 is given by

$$\begin{aligned} U_1(p_{n_1+1};P,Q) = p_{n_1+1} \Big(p_{n_1} \left(f_{Y,1}(p_{n_1};\underline{P},Q) + \alpha_{11}^Y \right) + (1-p_{n_1}) f_{Y,1}(p_{n_1};\underline{P},Q) \Big) \\ + (1-p_{n_1+1}) \Big(p_{n_1} f_{N,1}(p_{n_1};\underline{P},Q) + (1-p_{n_1}) \left(f_{N,1}(p_{n_1};\underline{P},Q) + \alpha_{11}^N \right) \Big) \ . \end{aligned}$$

Thus,

$$\begin{aligned} U_1(p_{n_1+1};P,Q) &= p_{n_1+1} \Big(f_{Y,1}(p_{n_1};\underline{P},Q) + p_{n_1} \alpha_{11}^Y \Big) \\ &\quad + (1-p_{n_1+1}) \Big(f_{N,1}(p_{n_1};\underline{P},Q) + (1-p_{n_1}) \alpha_{11}^N \Big) \ . \end{aligned}$$

By substituting the fitness functions in the previous identity, we obtain

$$\begin{aligned} U_1(p_{n_1+1};P,Q) &= p_{n_1+1} \Big(\omega_1^Y + \alpha_{11}^Y (\underline{P} + p_{n_1}) + \alpha_{12}^Y Q \Big) \\ &\quad + (1-p_{n_1+1}) \Big(\omega_1^N + \alpha_{11}^N (n_1 + 1 - (\underline{P} + p_{n_1})) + \alpha_{12}^N (n_2 - Q) \Big) \ . \end{aligned}$$

Hence,

$$U_1(p_{n_1+1}; P, Q) = p_{n_1+1} f_{Y,1}(p_{n_1+1}; P, Q) + (1 - p_{n_1+1}) f_{N,1}(p_{n_1+1}; P, Q) \ .$$

The proof follows similarly if we add one more individual $j = n_2 + 1$ with type t_2 and compute its utility function. □

A strategy $S^* : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ is a *(mixed) Nash equilibrium*, if

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$ with the property that $S^*(j) = S(j)$, for every individual $j \in I \setminus \{i\}$.

Lemma 5.2. *Let $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ be a mixed Nash equilibrium.*
(i): *If $0 < p_i < 1$, then $x = -A_{11}(P - p_i) - A_{12}Q + H(N, N)$.*
(ii): *If $0 < q_j < 1$, then $y = -A_{21}P - A_{22}(Q - q_j) + V(N, N)$.*

Hence, if $A_{11} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < p_{i_1} \neq p_{i_2} < 1$. Furthermore, if $A_{22} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < q_{j_1} \neq q_{j_2} < 1$.

Proof. Let $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ be a mixed Nash equilibrium. For every $p \in [0, 1]$, we have $U_1(p_i; P, Q) \geq U_1(p; P - p_i + p, Q)$. If $0 < p_i < 1$, we get

$$f_{Y,1}(p_i; P, Q) = f_{N,1}(p_i; P, Q)$$

which implies Lemma 5.2 (i). The proof of Lemma 5.2 (ii) follows similarly. □

The $(l_1, k_1, p; l_2, k_2, q)$ *mixed strategic set* is the set of all strategies $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ with the following properties:

(i): $l_1 = \#\{i \in I_1 : p_i = 1\}$ and $k_1 = \#\{i \in I_1 : p_i = p\}$;
(ii): $l_2 = \#\{j \in I_2 : q_j = 1\}$ and $k_2 = \#\{j \in I_2 : q_j = q\}$;
(iii): $n_1 - (l_1 + k_1) = \#\{i \in I_1 : p_i = 0\}$ and $n_2 - (l_2 + k_2) = \#\{j \in I_2 : q_j = 0\}$.

For $p, q \in \{0, 1\}$, we observe that the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set is equal to the $(l_1 + pk_1, l_2 + qk_2)$ pure strategic set.

Remark 1. By Lemma 5.2, supposing that $A_{11} \neq 0$ and $A_{22} \neq 0$, a mixed strategy S is a Nash equilibrium, if S is contained in some $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set.

Since individuals with the same type are identical, if a mixed strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ mix strategic set is a Nash equilibrium, then all the strategies in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set are Nash equilibria.

Definition 5.3. An $(l_1, k_1, p; l_2, k_2, q)$ *mixed Nash equilibrium (set)* is an $(l_1, k_1, p; l_2, k_2, q)$ strategic set whose strategies are Nash equilibria. The *(mixed) Nash domain* $\mathbf{N}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs (x, y) for which the $(l_1, k_1, p; l_2, k_2, q)$ strategic set is a mixed Nash equilibrium set.

An $(l_1, k_1, p; l_2, k_2, q)$ *strict mixed Nash equilibrium set* is a mixed Nash equilibrium set that does not contain pure strategies, i.e. $(p, q) \in [0, 1]^2 \setminus \{0, 1\}^2$. A *strict mixed Nash domain* $\mathbf{N}(l_1, k_1, p; l_2, k_2, q)$ is the mixed Nash domain of a strict mixed Nash equilibrium set.

The *mixed vector* $\vec{W}(l_1, k_1, p; l_2, k_2, q)$ is given by

$$\begin{aligned}\vec{W}(l_1, k_1, p; l_2, k_2, q) &= \vec{Z}(l_1, l_2) + p(\vec{Z}(k_1, 0) - (0, A_{21})) + q(\vec{Z}(0, k_2) - (A_{12}, 0)) \\ &= -l_1(A_{11}, A_{21}) - l_2(A_{12}, A_{22}) - p(k_1 A_{11}, (k_1+1)A_{21}) - q((k_2+1)A_{12}, k_2 A_{22}) \ .\end{aligned}$$

We observe that

$$\vec{W}(0, n_1 - 1, p; 0, n_2 - 1, q) = p\vec{H} + q\vec{V} \ .$$

Let $J_0 = (-\infty, 0)$ and $J_{n_1} = J_{n_2} = [0, +\infty)$.

Theorem 5.4. *Let $A_{11} > 0$ and $A_{22} > 0$. $\boldsymbol{N}(l_1, k_1, p; l_2, k_2, q) \neq \emptyset$ if, and only if, $l_1 = n_1 - k_1$, $l_2 = n_2 - k_2$, $k_1 \in \{0, n_1\}$ and $k_2 \in \{0, n_2\}$.*

(i): *For $l_1 \in \{0, n_1\}$ and $q \in (0, 1)$, the mixed Nash domain $\boldsymbol{N}(l_1, 0, 0; 0, n_2, q)$ is the semi-line*

$$\boldsymbol{N}(l_1, 0, 0; 0, n_2, q) = \{(x, 0) + \vec{W}(l_1, 0, 0; 0, n_2 - 1, q) : x \in J_{l_1}\} \ ;$$

(ii): *For $p \in (0, 1)$ and $l_2 \in \{0, n_2\}$, the mixed Nash domain $\boldsymbol{N}(0, n_1, p; l_2, 0, 0)$ is the semi-line*

$$\boldsymbol{N}(0, n_1, p; l_2, 0, 0) = \{(0, y) + \vec{W}(0, n_1 - 1, p; l_2, 0, 0) : y \in J_{l_2}\} \ ;$$

(iii): *For $p, q \in (0, 1)$, the mixed Nash domain $\boldsymbol{N}(0, n_1, p; 0, n_2, q)$ is the singleton*

$$\vec{W}(0, n_1 - 1, p; 0, n_2 - 1, q) + (H(N, N), V(N, N)) \ .$$

By Theorem 5.4, if $l_1 = n_1$, then the only strict mixed Nash equilibria are the ones presented in (i). Furthermore, if $l_1 = 0$, then the only strict mixed Nash equilibria are the ones presented in (ii) and (iii). Hence, if the individuals with a given type have a positive influence over the utility of the individuals with the same type, i.e. $A_{11} > 0$ and $A_{22} > 0$, then there are no mixed Nash equilibrium, unless all the individuals with the same type opt for a mixed strategy.

In Figure 1, we show the geometric interpretation of Theorem 5.4, with

$$\begin{aligned}\overrightarrow{W}_1 &= \overrightarrow{W}(0, 0, 0; 0, n_2 - 1, 1), \\ \overrightarrow{W}_2 &= \overrightarrow{W}(0, n_1 - 1, 1; 0, 0, 0), \\ Z_0 &= (H(N, N), V(N, N)), \\ Z_1 &= p\overrightarrow{W}_1 + q\overrightarrow{W}_2 + Z_0 = \overrightarrow{W}(0, n_1 - 1, p; 0, n_2 - 1, q) + Z_0.\end{aligned}$$

Proof. The mixed strategy $(0, 0, 0; 0, n_2, q)$ is a Nash equilibrium if, and only if, the following two inequalities hold

$$U_1(0; P, Q) \geq U_1(1; P + 1, Q) \ , \ U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') \ .$$

We note that

$$U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') \text{ ,if, and only if, } f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q) \ .$$

Hence, by Lemma 5.2,

$$y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

The mixed strategy $(n_1, 0, 0; 0, n_2, q)$ is a Nash equilibrium if, and only if, the following two inequalities hold

$$U_1(1; P, Q) \geq U_1(0; P - 1, Q) \ , \ U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') \ .$$

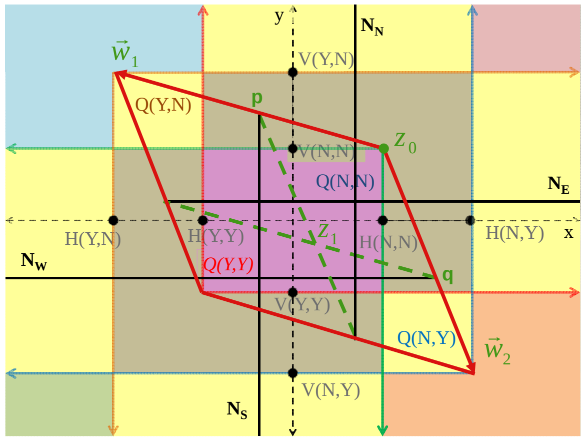

FIGURE 1. A decision tiling with mixed vectors, where $N_N = N(0,n_1,p;n_2,0,0)$, $N_S = N(0,n_1,p;0,0,0)$, $N_W = N(0,0,0;0,n_2,q)$ and $N_E = N(n_1,0,0;0,n_2,q)$.

We note that

$$U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \text{ , if, and only if, } f_{Y,2}(q;P,Q) = f_{N,2}(q;P,Q) \text{ .}$$

Hence, by Lemma 5.2,

$$y = -A_{21}P - A_{22}(Q-q) + V(N,N).$$

Thus, the proof of Theorem 5.4 (i) follows from rearranging the terms in the previous inequalities. The proof of Theorem 5.4 (ii) follows similarly to the proof of Theorem 5.4 (i). Let us prove Theorem 5.4 (iii).
The mixed strategy $(0,n_1,p;0,n_2,q)$ is a Nash equilibrium if, and only if, the following two inequalities hold

$$U_1(p;P,Q) \geq U_1(p';P-p+p',Q) \text{ , } U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \text{ .}$$

Therefore,

$$f_{Y,1}(p;P,Q) = f_{N,1}(p;P,Q) \text{ and } f_{Y,2}(q;P,Q) = f_{N,2}(q;P,Q) \text{ .}$$

Thus, by Lemma 5.2,

$$x = -A_{11}(P-p_i) - A_{12}Q + H(N,N) \text{ and } y = -A_{21}P - A_{22}(Q-q) + V(N,N).$$

Hence, Theorem 5.4 (iii) follows from rearranging the terms in the previous inequalities. ☐

Let $J_0(2) = (-\infty,0)$ and $J_{n_2}(2) = [A_{22},+\infty)$. Let $J_{l_2}(2) = [A_{22},0]$, for $l_2 \in \{0,\ldots,n_2\} \setminus \{0,n_2\}$.

Theorem 5.5. *Let $A_{11} > 0$ and $A_{22} \leq 0$. $\boldsymbol{N}(l_1,k_1,p;l_2,k_2,q) \neq \emptyset$ if, and only if, $l_1 = n_1 - k_1$ and $k_1 \in \{0,n_1\}$.*

(i): *For $l_1 \in \{0,n_1\}$, $k_2 \geq 1$ and $q \in (0,1)$, the mixed Nash domain $\boldsymbol{N}(l_1,0,0;l_2,k_2,q)$ is the semi-line*

$$\boldsymbol{N}(l_1,0,0;l_2,k_2,q) = \{(x,0) + \vec{W}(l_1,0,0;l_2,k_2-1,q) : x \in J_{l_1}\} \text{ ;}$$

(ii): *For $p \in (0,1)$ and $l_2 \in \{0,\ldots,n_2\}$, the mixed Nash domain $\boldsymbol{N}(0,n_1,p;l_2,0,0)$ is the semi-line*

$$\boldsymbol{N}(0,n_1,p;l_2,0,0) = \{(0,y) + \vec{W}(0,n_1-1,p;l_2,0,0) : y \in J_{l_2}(2)\} \ ;$$

(iii): *For $p,q \in (0,1)$, the mixed Nash domain $\boldsymbol{N}(0,n_1,p;l_2,k_2,q)$ is the singleton*

$$\vec{W}(0,n_1-1,p;l_2,k_2-1,q) + (H(N,N),V(N,N)) \ .$$

By Theorem 5.5, if $l_1 = n_1$, then the only strict mixed Nash equilibria are the ones presented in (i). Furthermore, if $l_1 = 0$, then the only strict mixed Nash equilibria are the ones presented in (ii) and (iii). Hence, if the individuals with type t_1 have a positive influence over the utility of the individuals with the same type, i.e. $A_{11} > 0$, then, for every Nash equilibrium all the individuals with the type t_1 opt for the same strategy either pure or mixed.

Proof. The mixed strategy $(0,0,0;l_2,k_2,q)$ is a Nash equilibrium if the following two inequalities hold

$$U_1(0;P,Q) \geq U_1(1;P+1,Q) \ , \ U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \ .$$

We note that

$$U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \text{ ,if, and only if, } f_{Y,2}(q;P,Q) = f_{N,2}(q;P,Q) \ .$$

Hence, by Lemma 5.2,

$$y = -A_{21}P - A_{22}(Q-q) + V(N,N).$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1;P,Q) \geq U_2(0;P,Q-1)$ and (b) if $n_2 > l_2 + k_2$ then $U_2(0;P,Q) \geq U_2(1;P,Q+1)$. Hence, the proof follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$.
The mixed strategy $(n_1,0,0;l_2,k_2,q)$ is a Nash equilibrium if the following two inequalities hold

$$U_1(1;P,Q) \geq U_1(0;P-1,Q) \ , \ U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \ .$$

We note that

$$U_2(q;P,Q) \geq U_2(q';P,Q-q+q') \text{ ,if, and only if, } f_{Y,2}(q;P,Q) = f_{N,2}(q;P,Q) \ .$$

Hence, by Lemma 5.2,

$$y = -A_{21}P - A_{22}(Q-q) + V(N,N).$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1;P,Q) \geq U_2(0;P,Q-1)$ and (b) if $n_2 > l_2 + k_2$ then $U_2(0;P,Q) \geq U_2(1;P,Q+1)$. Hence, the proof follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$.
The mixed strategy $(0,n_1,p;l_2,0,0)$ is a Nash equilibrium if the following inequality holds

$$U_1(p;P,Q) \geq U_1(p';P-p+p',Q).$$

We note that

$$U_1(p;P,Q) \geq U_1(p';P-p+p',Q) \text{ ,if, and only if, } f_{Y,1}(p;P,Q) = f_{N,1}(p;P,Q) \ .$$

Hence, by Lemma 5.2,

$$x = -A_{11}(P-p_i) - A_{12}Q + H(N,N).$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1;P,Q) \geq U_2(0;P,Q-1)$ and (b) if $n_2 > l_2$ then $U_2(0;P,Q) \geq U_2(1;P,Q+1)$. Hence, the proof of Theorem 5.5 (ii) follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$. Let us

prove Theorem 5.5 (iii).
The mixed strategy $(0, n_1, p; l_2, k_2 + 1, q)$ is a Nash equilibrium if the following inequalities hold

$$U_1(p; P, Q) \geq U_1(p'; P - p + p', Q) \ , \ U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') \ .$$

Therefore,

$$f_{Y,1}(p; P, Q) = f_{N,1}(p; P, Q) \text{ and } f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q) \ .$$

Thus, by Lemma 5.2,

$$x = -A_{11}(P - p_i) - A_{12}Q + H(N, N) \text{ and } y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$ and (b) if $n_2 > l_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 5.5 (iii) follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$. □

Let $J_0(1) = (-\infty, 0)$ and $J_{n_1}(1) = [A_{11}, +\infty)$. Let $J_{l_1}(1) = [A_{11}, 0]$, for $l_1 \in \{0, \ldots, n_1\} \setminus \{0, n_1\}$.

Theorem 5.6. *Let $A_{11} \leq 0$ and $A_{22} \leq 0$.*

(i): *For $l_1 \in \{0, \ldots, n_1\}$, $k_2 \geq 1$ and $q \in (0, 1)$, the mixed Nash domain $\boldsymbol{N}(l_1, 0, 0; l_2, k_2, q)$ is the semi-line*

$$\boldsymbol{N}(l_1, 0, 0; l_2, k_2, q) = \{(x, 0) + \vec{W}(l_1, 0, 0; l_2, k_2 - 1, q) : x \in J_{l_1}(1)\} \ ;$$

(ii): *For $p \in (0, 1)$, $k_1 \geq 1$ and $l_2 \in \{0, \ldots, n_2\}$, the mixed Nash domain $\boldsymbol{N}(l_1, k_1, p; l_2, 0, 0)$ is the semi-line*

$$\boldsymbol{N}(l_1, k_1, p; l_2, 0, 0) = \{(0, y) + \vec{W}(l_1, k_1 - 1, p; l_2, 0, 0) : y \in J_{l_2}(2)\} \ ;$$

(iii): *For $k_1 \geq 1$, $k_2 \geq 1$ and $p, q \in (0, 1)$, the mixed Nash domain $\boldsymbol{N}(l_1, k_1, p; l_2, k_2, q)$ is the singleton*

$$\vec{W}(l_1, k_1 - 1, p; l_2, k_2 - 1, q) + (H(N, N), V(N, N)) \ .$$

By Theorem 5.6, if the individuals with a given type have a non-positive influence over the utility of the individuals with the same type, i.e. $A_{11} \leq 0$ and $A_{22} \leq 0$, then for every $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set there are relative preferences for which $(l_1, k_1, p; l_2, k_2, q)$ is a Nash equilibrium set.

In Figures 2 and 3, we show the geometric interpretation of Theorem 5.6, with

$$\begin{aligned}
Z_0 &= \overrightarrow{Z}(1, 1) + (H(N, N), V(N, N)), \\
\overrightarrow{u}_1 &= \overrightarrow{Z}(0, 0) - (0, A_{21}) = -(0, A_{21}), \\
\overrightarrow{u}_2 &= \overrightarrow{Z}(0, 0) - (A_{12}, 0) = -(A_{12}, 0), \\
\overrightarrow{u} &= p_1 \overrightarrow{u}_1 + q_1 \overrightarrow{u}_2, \\
Z_1 &= \overrightarrow{u} + Z_0 = \overrightarrow{W}(1, 2, p_2; 1, 2, q_2) + Z_0
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{w}_1 &= \overrightarrow{Z}(1, 0) - (0, A_{21}) = -(A_{11}, A_{21}), \\
\overrightarrow{w}_2 &= \overrightarrow{Z}(0, 1) - (A_{12}, 0) = -(A_{12}, A22), \\
\overrightarrow{w} &= p_2 \overrightarrow{w}_1 + q_2 \overrightarrow{w}_2, \\
Z_2 &= \overrightarrow{w} + Z_0 = \overrightarrow{W}(1, 1, p_1; 1, 1, q_1) + Z_0.
\end{aligned}$$

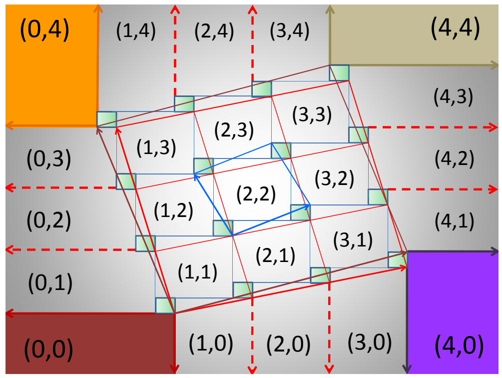

FIGURE 2. A decision tiling with mixed vectors. The influence matrix is given by $A_{11} = -2$, $A_{12} = 1/2$, $A_{22} = -1/2$ and $A_{21} = -2$ and $(n_1, n_2) = (4, 4)$.

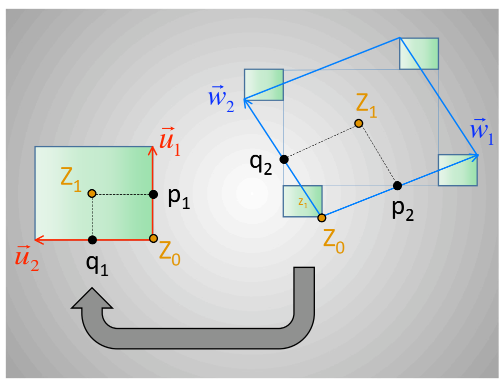

FIGURE 3. Zooming Figure 2. The Nash equilibria $\{Z_1\} = \mathbf{N}(1, 1, p_1; 1, 1, q_1)$ and $\{Z_2\} = \mathbf{N}(1, 2, p_2; 1, 2, q_2)$.

Proof. The mixed strategy $(l_1, 0, 0; l_2, k_2, q)$ is a Nash equilibrium if the following inequality holds

$$U_2(q; P, Q) \geq U_2(q'; P, Q - q + q').$$

Hence,

$$f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q).$$

Therefore, by Lemma 5.2,

$$y = -A_{21}P - A_{22}(Q - q_j) + V(N, N).$$

Furthermore, (a) if $l_1 \geq 1$ then $U_1(1; P, Q) \geq U_1(0; P - 1, Q)$; (b) if $n_1 > l_1$ then $U_1(0; P, Q) \geq U_1(1; P + 1, Q)$; (c) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$; and (d) if $n_2 > l_2 + k_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 5.6 (i) follows from rearranging the terms in the previous inequalities and noting that $A_{11} \leq 0$ and $A_{22} \leq 0$. The proof of Theorem 5.6 (ii) follows similarly.Let us prove Theorem 5.6 (iii).

The mixed strategy $(l_1, k_1, p; l_2, k_2, q)$ is a Nash equilibrium if the following inequalities hold

$$U_1(p; P, Q) \geq U_1(p'; P - p + p', Q)\ ,\ U_2(q; P, Q) \geq U_2(q'; P, Q - q + q')\ .$$

Hence,

$$f_{Y,1}(p; P, Q) = f_{N,1}(p; P, Q) \text{ and } f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q).$$

Therefore, by Lemma 5.2,

$$x = -A_{11}(P - p) - A_{12}Q + H(N, N) \text{ and } y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

Furthermore, (a) if $l_1 \geq 1$ then $U_1(1; P, Q) \geq U_1(0; P - 1, Q)$; (b) if $n_1 > l_1 + k_1$ then $U_1(0; P, Q) \geq U_1(1; P + 1, Q)$; (c) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$; and (d) if $n_2 > l_2 + k_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 5.6 (iii) follows by rearranging the terms in the previous inequalities and noting that $A_{11} \leq 0$ and $A_{22} \leq 0$. □

Lemma 5.7. *Let S be a strategy given by $S(i) = p_i$, for $i \in I_1$, and $S(j) = q_j$, for $j \in I_2$.*

(i): *If $A_{11} = 0$, the strategy S at (x, y) is a Nash equilibrium if, and only if, there is $0 < q < 1$, such that*

$$\#\{j \in I_2 : q_j = 1\} = l_2;$$
$$\#\{j \in I_2 : q_j = q\} = k_2 \text{ (k_2 might be 0)};$$
$$\#\{j \in I_2 : q_j = 0\} = n_2 - (l_2 + k_2)$$

and

$$(x, y) \in \boldsymbol{N}(0, n_1, P/n_1; l_2, k_2, q).$$

(ii): *If $A_{11} \neq 0$ and $A_{22} = 0$, the strategy S at (x, y) is a Nash equilibrium if, and only if, there is $0 < p < 1$, such that*

$$\#\{i \in I_1 : p_i = 1\} = l_1;$$
$$\#\{i \in I_1 : p_i = q\} = k_1 \text{ (k_1 might be 0)};$$
$$\#\{i \in I_1 : p_i = 0\} = n_1 - (l_1 + k_1)$$

and

$$(x, y) \in \boldsymbol{N}(l_1, k_1, p; 0, n_2, Q/n_2).$$

(iii): *If $A_{11} = 0$ and $A_{22} = 0$, the strategy S at (x, y) is a Nash equilibrium if, and only if*

$$(x, y) \in \boldsymbol{N}(0, n_1, P/n_1; 0, n_2, Q/n_2).$$

Proof. It follows from putting together Lemma 5.2 with Theorems 5.5 and 5.6. □

In the following remark, we observe, for a mixed Nash equilibria, who receives higher utility between (i) the individuals who decide Y, (ii) the individuals who decide N, and (iii) the individuals who decide based on probability.

Remark 2. Let $(l_1, k_1, p; l_2, k_2, q)$ be a mixed Nash equilibria.

(i) If $\alpha_{11}^{Y} \leq 0$, then $U_1(1,P,Q) \geq U_1(p,P,Q)$;

(ii) If $\alpha_{11}^{Y} \geq 0$, then $U_1(1,P,Q) \leq U_1(p,P,Q)$;

(iii) If $\alpha_{11}^{N} \leq 0$, then $U_1(0,P,Q) \leq U_1(p,P,Q)$;

(iv) If $\alpha_{11}^{N} \geq 0$, then $U_1(0,P,Q) \geq U_1(p,P,Q)$.

6. **Replicator Dynamics.** Recall that $I = I_1 \sqcup I_2$ and that a strategy $S : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$ associates to each individual $i \in \mathbf{I_1}$ the probability $p_i = S(i)$ to decide $Y \in \mathbf{D}$ and to each individual $j \in \mathbf{I_2}$ the probability $q_j = S(j)$ to decide $Y \in \mathbf{D}$. Recall that $P = \sum_{i=1}^{n_1} p_i$,$Q = \sum_{j=1}^{n_2} q_j$, $P_i = P - p_i$ and $Q_j = Q - q_j$. The *replicator dynamics* $\dot{S} = G(S;x,y)$ is given by

$$\begin{cases} \frac{\dot{p}_i}{p_i} = f_{Y,1}(p_i;P,Q) - U_1(p_i;P,Q) = (1-p_i)\Big(f_{Y,1}(p_i;P,Q) - f_{N,1}(p_i;P,Q)\Big) \ , \\ \\ \frac{\dot{q}_j}{q_j} = f_{Y,2}(q_j;P,Q) - U_2(q_j;P,Q) = (1-q_j)\Big(f_{Y,2}(q_j;P,Q) - f_{N,2}(q_j;P,Q)\Big) \ . \end{cases}$$

Hence, the *replicator dynamics* $\dot{S} = G(S;x,y)$ can be rewritten as

$$\begin{cases} \dot{p}_i = p_i(1-p_i)\left(P_iA_{11} + QA_{12} + x - H(N,N)\right) \ , \ i \in \{1,\ldots,n_1\} \ , \\ \\ \dot{q}_j = q_j(1-q_j)(Q_jA_{22} + PA_{21} + y - V(N,N)) \ , \ j \in \{1,\ldots,n_2\} \ . \end{cases}$$

We observe that

(i) if $p_i(0) < p_j(0)$, then $p_i(t) < p_j(t)$;

(ii) if $p_i(0) = p_j(0)$, then $p_i(t) = p_j(t)$;

(iii) if $q_i(0) < q_j(0)$, then $q_i(t) < q_j(t)$;

(iv) if $q_i(0) = q_j(0)$, then $q_i(t) = q_j(t)$;

for every $t \in \mathbb{R}$.

A strategy $S : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$ is a *dynamical equilibrium* if $G(S;x,y) = 0$, i.e.

$$\begin{cases} p_i(1-p_i)\left(P_iA_{11} + QA_{12} + x - H(N,N)\right) = 0 \ , \ i \in \{1,\ldots,n_1\} \ , \\ \\ q_j(1-q_j)(Q_jA_{22} + PA_{21} + y - V(N,N)) = 0 \ , \ j \in \{1,\ldots,n_2\} \ . \end{cases}$$

The coefficients of the *linearized replicator dynamics* $DG(S;x,y)$ are

$$\begin{cases} \partial\dot{p}_i/\partial p_i = (1-2p_i)(P_iA_{11} + QA_{12} + x - H(N,N)) \ , \\ \partial\dot{p}_i/\partial p_j = p_i(1-p_i)A_{11} \ , \ i \neq j \\ \partial\dot{p}_i/\partial q_j = p_i(1-p_i)A_{12} \ , \\ \partial\dot{q}_j/\partial q_j = (1-2q_j)(Q_jA_{22} + PA_{21}y + y - V(N,N)) \ , \\ \partial\dot{q}_j/\partial q_i = q_j(1-q_j)A_{22} \ , \ j \neq i \\ \partial\dot{q}_j/\partial p_i = q_j(1-q_j)A_{21} \ . \end{cases}$$

An equilibrium strategy S is *(strongly) stable* if all the eigenvalues of the linearized replicator dynamics $DG(S;x,y)$ have real parts that are negative. An equilibrium strategy S is *(strongly) unstable* if there is at least one eigenvalue of the linearized replicator dynamics $DG(S;x,y)$ with positive real part.

Lemma 6.1. *Let $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ be a dynamical equilibrium of the replicator dynamics.*

(i): *If $0 < p_i < 1$, then $x = -A_{11}(P - p_i) - A_{12}Q + H(N, N)$.*
(ii): *If $0 < q_j < 1$, then $y = -A_{21}P - A_{22}(Q - q_j) + V(N, N)$.*

Hence, if $A_{11} \neq 0$, then there is not a dynamical equilibrium with the property that $0 < p_{i_1} \neq p_{i_2} < 1$. Furthermore, if $A_{22} \neq 0$, then there is not a dynamical equilibrium with the property that $0 < q_{j_1} \neq q_{j_2} < 1$.

Proof. The proof is analogous to the proof of Lemma 5.2. □

Hence, assuming that $A_{11} \neq 0$ and $A_{22} \neq 0$, the dynamical equilibria of the replicator dynamics are contained in the union of all $(l_1, k_1, p; l_2, k_2, q)$ strategic sets, where $p, q \in [0, 1]$, $0 \leq l_1 + k_1 \leq n_1$ and $0 \leq l_2 + k_2 \leq n_2$. Thus, to find and study the dynamical equilibria we introduce the following notation:

- $v[1l] = (v_1^{1l}, \ldots, v_{l_1}^{1l})$,$v[1m] = (v_1^{1m}, \ldots, v_{n_1-(l_1+k_1)}^{1m})$ and $v[1r] = (v_1^{1r}, \ldots, v_{k_1}^{1r})$;
- $v[2l] = (v_1^{2l}, \ldots, v_{l_2}^{2l})$, $v[2m] = (v_1^{2m}, \ldots, v_{n_2-(l_2+k_2)}^{2m})$ and $v[2r] = (v_1^{2r}, \ldots, v_{k_2}^{2r})$.

Let us define the vectors $v[1] \in \mathbb{R}^{n_1}$, $v[2] \in \mathbb{R}^{n_2}$ and $v \in \mathbb{R}^{n_1+n_2}$ as follows:

$$v[1] = (v[1l], v[1m], v[1r]) \ , \ v[2] = (v[2l], v[2m], v[2r]) \text{ and } v = (v[1], v[2]) \ .$$

Let us define $V[1]$ and $V[2]$ as follows:

- $V[1] = \sum_{i=1}^{l_1} v_i^{1l} + \sum_{j=1}^{n_1-(l_1+k_1)} v_j^{1m} + \sum_{k=1}^{k_1} v_k^{1r}$ and
- $V[2] = \sum_{i=1}^{l_2} v_i^{21} + \sum_{j=1}^{n_2-(l_2+k_2)} v_j^{2m} + \sum_{k=1}^{k_2} v_k^{2r}$.

In this notation, the *replicator dynamics* $DG(v; x, y)$ are given by the following $n_1 + n_2$ ODE:

$$\begin{cases} \dot{v}_{i_1}^{1m_1} = v_{i_1}^{1m_1}\left(1 - v_{i_1}^{1m_1}\right)\left(\left(V[1] - v_{i_1}^{1m_1}\right)A_{11} + V[2]A_{12} + x - H(N, N)\right) \ , i_1 \in I_1 \\ \\ \dot{v}_{i_2}^{2m_2} = v_{i_2}^{2m_2}\left(1 - v_{i_2}^{2m_2}\right)\left(\left(V[2] - v_{i_2}^{2m_2}\right)A_{22} + V[1]A_{21} + y - V(N, N)\right) \ , i_2 \in I_2 \end{cases}$$

where $m_1, m_2 \in \{l, m, r\}$ and $(x, y) \in \mathbb{R}^2$.

Definition 6.2. The $(l_1, k_1, p; l_2, k_2, q)$ *canonical strategy* is defined as follows:

- for all $i \in \{1, \ldots, l_1\}$, $j \in \{1, \ldots, n_1 - (l_1 + k_1)\}$ and $k \in \{1, \ldots, k_1\}$
$$v_i^{1l} = 1 \ , \ v_j^{1m} = 0 \ , \text{ and } v_k^{1r} = p \ ;$$
- for all $i \in \{1, \ldots, l_2\}$, $j \in \{1, \ldots, n_2 - (l_2 + k_2)\}$ and $j \in \{1, \ldots, k_2\}$
$$v_i^{2l} = 1 \ , \ v_j^{2m} = 0 \ , \text{ and } v_k^{2r} = q \ .$$

The linearized replicator dynamics $DG = DG(l_1, k_1, p; l_2, k_2, q; x, y)$ at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is given by the matrix

$$DG = \begin{pmatrix} f[1l, 1l] & 0 & 0 & 0 & 0 & 0 \\ \\ 0 & f[1m, 1m] & 0 & 0 & 0 & 0 \\ f[1r, 1l] & f[1r, 1m] & f[1r, 1r] & f[1r, 2l] & f[1r, 2m] & f[1r, 2r] \\ 0 & 0 & 0 & f[2l, 2l] & 0 & 0 \\ 0 & 0 & 0 & 0 & f[2m, 2m] & 0 \\ f[2r, 1l] & f[2r, 1m] & f[2r, 1r] & f[2r, 2l] & f[2r, 2m] & f[2r, 2r] \end{pmatrix} ,$$

where the coefficients are matrices with the following coordinates:

- for all $i \in \{1, \ldots, l_1\}$,

$$f_{ii}[1l, 1l] = -(l_1 - 1 + k_1 p)A_{11} - (l_2 + k_2 q)A_{12} - x + H(N, N) \ ;$$

- for all $l \in \{1, \ldots, n_1 - (l_1 + k_1)\}$,

$$f_{ll}[1m, 1m] = (l_1 + k_1 p)A_{11} + (l_2 + k_2 q)A_{12} + x - H(N, N) \ ;$$

- for all $i, j \in \{1, \ldots, l_1\}$, with $i \neq j$, and for all $l, k \in \{1, \ldots, n_1 - (l_1 + k_1)\}$, with $l \neq k$,

$$f_{ij}[1l, 1l] = f_{l,k}[1m, 1m] = 0 \ ;$$

- for all $i \in \{1, \ldots, k_1\}$,

$$f_{ii}[1r, 1r] = (1 - 2p)((l_1 + (k_1 - 1)p)A_{11} + (l_2 + k_2 q)A_{12} + x - H(N, N)) \ ;$$

- for all $i, j \in \{1, \ldots, k_1\}$, with $i \neq j$,

$$f_{ij}[1r, 1r] = p(1 - p)A_{11} \ ;$$

- for all $i \in \{1, \ldots, k_1\}$, $j \in \{1, \ldots, l_2\}$, $k \in \{1, \ldots, n_2 - (l_2 + k_2)\}$ and $l \in \{1, \ldots, k_2\}$,

$$f_{ij}[1r, 2l] = f_{ik}[1r, 2m] = f_{il}[1r, 2r] = p(1 - p)A_{12} \ ;$$

- for all $i \in \{1, \ldots, l_2\}$,

$$f_{ii}[2l, 2l] = -(l_2 - 1 + k_2 q)A_{22} - (l_1 + k_1 p)A_{21} - y + V(N, N) \ ;$$

- for all $l \in \{1, \ldots, n_2 - (l_2 + k_2)\}$,

$$f_{ll}[2m, 2m] = (l_2 + k_2 q)A_{22} + (l_1 + k_1 p)A_{21} + y - V(N, N) \ ;$$

- for all $i, j \in \{1, \ldots, l_2\}$, with $i \neq j$, and for all $l, k \in \{1, \ldots, n_2 - (l_2 + k_2)\}$, with $l \neq k$,

$$f_{ij}[2l, 2l] = f_{l,k}[2m, 2m] = 0 \ ;$$

- for all $i \in \{1, \ldots, k_2\}$,

$$f_{ii}[2r, 2r] = (1 - 2q)((l_2 + (k_2 - 1)q)A_{22} + (l_1 + k_1 p)A_{21} + y - V(N, N)) \ ;$$

- for all $i, j \in \{1, \ldots, k_2\}$, with $i \neq j$,

$$f_{ij}[2r, 2r] = q(1 - q)A_{22} \ ;$$

- for all $i \in \{1, \ldots, k_2\}$, $j \in \{1, \ldots, l_1\}$, $k \in \{1, \ldots, n_1 - (l_1 + k_1)\}$ and $l \in \{1, \ldots, k_1\}$,

$$f_{ij}[2r, 1l] = f_{ik}[2r, 1m] = f_{il}[2r, 1r] = q(1 - q)A_{21} \ .$$

We observe that for some pair $(x, y) \in \mathbb{R}^2$, if the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is an equilibrium of the replicator dynamics, then every strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set is an equilibrium of the replicator dynamics. Furthermore, the eigenvectors of the linearized replicator dynamics at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy coincide with the eigenvectors of the linearized replicator dynamics at any strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set, up to permutation of coordinates. Hence, the eigenvalues of the linearized replicator dynamics at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy coincide with the eigenvalues of the linearized replicator dynamics at any strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set.

Definition 6.3. The *equilibria domain* $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are equilibria of the replicator dynamics. The *(strongly) stable domain* $\mathbf{S}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are (strongly) stable equilibria of the replicator dynamics. The *(strongly) unstable domain* $\mathbf{U}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are (strongly) unstable equilibria of the replicator dynamics.

We observe that the stable domains are contained in the Nash domains and the Nash domains are contained in the equilibria domains, i.e.

$$\mathbf{S}(l_1, k_1, p; l_2, k_2, q) \subseteq \mathbf{N}(l_1, k_1, p; l_2, k_2, q) \subseteq \mathbf{E}(l_1, k_1, p; l_2, k_2, q) \ .$$

For the (l_1, l_2) (pure) strategic set, the dynamic equilibria set coincides with $\mathbb{R}^2$, i.e.

$$\mathbf{E}(l_1, l_2) = \mathbb{R}^2 \ .$$

The following geometric relation associates to every pair $(x, y) \in \mathbb{R}^2$ a unique pair $(p, q) \in \mathbb{R}^2$ and vice-versa:

$$\begin{aligned} (x, y) &= (H(N, N), V(N, N)) + \vec{Z}(l_1, l_2) + (pA_{11}, qA_{22}) \\ &= (H(N, N) - l_1 A_{11} - l_2 A_{12} + pA_{11}, V(N, N) - l_2 A_{22} - l_1 A_{21} + qA_{22}) \ . \end{aligned}$$

Theorem 6.4. *The eigenvalues of* $DG(l_1, l_2; x, y) = DG(l_1, 0, 0; l_2, 0, 0; x, y)$ *are*

$$\lambda[1l] = A_{11}(1 - p) \ , \ \lambda[1m] = A_{11} p \ , \ \lambda[2l] = A_{22}(1 - q) \ , \ \lambda[2m] = A_{22} q \ . \quad (2)$$

The eigenspaces of $DG(l_1, l_2; x, y)$ *are*

$$\begin{aligned} E(\lambda[1l]) &= \{v : (v[1m], v[2]) = 0\} \ , \ E(\lambda[1m]) = \{v : (v[1l], v[2]) = 0\}, \\ E(\lambda[2l]) &= \{v : (v[1], v[2m]) = 0\} \ , \ E(\lambda[2m]) = \{v : (v[1], v[2l]) = 0\} \ . \quad (3) \end{aligned}$$

Furthermore,

$$\boldsymbol{S}(l_1, l_2) = int(\boldsymbol{N}(l_1, l_2)) \ \text{and} \ \boldsymbol{U}(l_1, l_2) \subset \mathbb{R}^2 \setminus \boldsymbol{N}(l_1, l_2).$$

Putting together Lemma 4.3 and Theorem 6.4, we obtain the following:

- if (a) $A_{11} \geq 0$ and $l_1 \in \{1, \ldots, n_1 - 1\}$, or if (b) $A_{22} \geq 0$ and $l_2 \in \{1, \ldots, n_2 - 1\}$, or if (c) $A_{11} \geq 0$ and $A_{22} \geq 0$ and $(l_1, l_2) \in \{1, \ldots, n_1 - 1\} \times \{1, \ldots, n_2 - 1\}$, then
$$\mathbf{S}(l_1, l_2) = \text{int}(\mathbf{N}(l_1, l_2)) = \emptyset.$$
- if (a) $A_{11} < 0$ and $l_2 \in \{0, n_2\}$, or if (b) $A_{22} < 0$ and $l_1 \in \{0, n_1\}$, or if (c) $A_{11} < 0$ and $A_{22} < 0$ and $(l_1, l_2) \in \{(0, 0), (n_1, 0), (0, n_2), (n_1, n_2)\}$, then
$$\mathbf{S}(l_1, l_2) = \text{int}(\mathbf{N}(l_1, l_2)) \neq \emptyset.$$

Proof. The coefficients of the linearized replicator dynamics $DG = DG(l_1, 0, 0; l_2, 0, 0; x, y)$ are the following: If $i \in \{1, \ldots, l_1\}$, then

$$f_{ii}[1l, 1l] = -(l_1 - 1)A_{11} - l_2 A_{12} - x + H(N, N) = A_{11}(1 - p) \ .$$

If $i \in \{1, \ldots, n_1 - l_1\}$, then

$$f_{ii}[1m, 1m] = l_1 A_{11} + l_2 A_{12} + x - H(N, N) = A_{11} p \ .$$

If $j \in \{1, \ldots, l_2\}$, then

$$f_{jj}[2l, 2l] = -(l_2 - 1)A_{22} - l_1 A_{21} - y + V(N, N) = A_{22}(1 - q) \ .$$

If $j \in \{1, \ldots, n_2 - l_2\}$, then

$$f_{jj}[2m, 2m] = l_2 A_{22} + l_1 A_{21} + y - V(N, N) = A_{22} q \ .$$

Furthermore, all the other coefficients of DG are equal to 0. Hence, DG is a diagonal matrix. The eigenvalues and the eigenspaces of DG are the ones presented in (2) and (3). Hence, the proof of the second part of Theorem 6.4 follows from applying Lemma 4.3. □

From Theorems 5.4, 5.5 and 5.6, the equilibrium domain $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the following:

(i) For $l_1 \in \{0, \ldots, n_1\}$ and $q \in (0, 1)$, the equilibrium domain $\mathbf{E}(l_1, 0, 0; l_2, k_2, q)$ is the line

$$\mathbf{E}(l_1, 0, 0; l_2, k_2, q) = \{(x, 0) + \vec{W}(l_1, 0, 0; l_2, k_2 - 1, q) : x \in \mathbb{R}\} \ ;$$

(ii) For $p \in (0, 1)$ and $l_2 \in \{0, \ldots, n_2\}$, the equilibrium domain $\mathbf{E}(l_1, k_1, p; l_2, 0, 0)$ is the line

$$\mathbf{E}(l_1, k_1, p; l_2, 0, 0) = \{(0, y) + \vec{W}(l_1, k_1 - 1, p; l_2, 0, 0) : y \in \mathbb{R}\} \ ;$$

(iii) For $p, q \in (0, 1)$, the equilibrium domain $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the singleton

$$\vec{W}(l_1, k_1 - 1, p; l_2, k_2 - 1, q) + (H(N, N), V(N, N)) \ .$$

Define the matrix $M(k_1, k_2) = M(l_1, k_1, p; l_2, k_2, q; x, y)$ as follows:

- if $k_1 \geq 1$ and $k_2 \geq 1$, then

$$M(k_1, k_2) = \begin{pmatrix} (k_1 - 1)p(1 - p)A_{11} & k_1 q(1 - q)A_{21} \\ k_2 p(1 - p)A_{12} & (k_2 - 1)q(1 - q)A_{22} \end{pmatrix} \ ,$$

- if $k_1 = 0$ and $k_2 \geq 1$, then

$$M(0, k_2) = ((k_2 - 1)q(1 - q)A_{22}) \ ,$$

- if $k_1 \geq 1$ and $k_2 = 0$, then

$$M(k_1, 0) = ((k_1 - 1)p(1 - p)A_{11}) \ .$$

The following relation associates to each pair $(x, y) \in \mathbb{R}^2$ a unique pair $(p, q) \in \mathbb{R}^2$ and vice-versa:

$$(x, y) - (H(N, N), V(N, N)) = \\ -\left((l_1 + p(k_1 - 1)) A_{11} + (l_2 + qk_2) A_{12}, (l_2 + q(k_2 - 1)) A_{22} + (l_1 + pk_1) A_{21} \right) \ .$$

Theorem 6.5. *Let $DG = DG(l_1, k_1, p; l_2, k_2, q; x, y)$ and*

$$(p, q) \in [0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The eigenvalues of DG are the following:

- $\lambda[1l] = (1 - p)A_{11}$, *with algebraic dimension l_1;*
- $\lambda[1m] = pA_{11}$, *with algebraic dimension $n_1 - (l_1 + k_1)$;*
- $\lambda[2l] = (1 - q)A_{22}$, *with algebraic dimension l_2;*
- $\lambda[2m] = qA_{22}$, *with algebraic dimension $n_1 - (l_2 + k_2)$;*
- $\lambda[1r] = -p(1 - p)A_{11}$, *with algebraic dimension $k_1 - 1$;*
- $\lambda[2r] = -q(1 - q)A_{22}$, *with algebraic dimension $k_2 - 1$;*
- *The eigenvalues of the matrix $M(l_1, k_1, p; l_2, k_2, q; x, y)$.*

Furthermore, $S(l_1, k_1, p; l_2, k_2, q) = \emptyset$.

Theorem 6.5 also implies the following unstability properties for the equilibria $E(l_1, k_1, p; l_2, k_2, q; x, y)$.

- Let $A_{11} > 0$ and $A_{22} > 0$. The sum of the eigenvalues of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ is greater than zero. Hence, $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$.
- Let $A_{11} > 0$ and $A_{22} < 0$. (a) If $k_2 \geq 2$ then $\lambda[2r] > 0$. (b) If $k_1 < n_1$ then $\lambda[1l]$ or $\lambda[1m]$ has real part greater than zero. (c) If $k_1 = n_1$ and $k_2 = 0$ then the eigenvalue of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ has real part greater than zero. (d) If $k_1 = n_1$ and $k_2 = 1$ then the sum of the eigenvalues of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ is greater than zero. Hence, $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$.
- Let $A_{11} < 0$ and $A_{22} < 0$.
  - If $k_1 \geq 2$ or $k_2 \geq 2$, then $\lambda[1r]$ or $\lambda[2r]$ is positive. Hence, $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$.
  - If $k_1 = 1$ and $k_2 = 1$, then the eigenvalues $\lambda_\pm$ of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ are
  $$\lambda_\pm^2 = p(1-p)q(1-q)A_{12}A_{21}.$$
  Hence, (a) if $A_{12}A_{21} > 0$ then $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$; (b) if $A_{12}A_{21} \leq 0$ then
  $$U(l_1, k_1, p; l_2, k_2, q) = S(l_1, k_1, p; l_2, k_2, q) = \emptyset,$$
  i.e. the mixed Nash equilibria is neither strongly stable nor strongly unstable.
  - if $k_1 = 1$ and $k_2 = 0$, then $M(l_1, k_1, p; l_2, k_2, q; x, y) = (0)$. Hence,
  $$U(l_1, k_1, p; l_2, k_2, q) = S(l_1, k_1, p; l_2, k_2, q) = \emptyset,$$
  i.e. the mixed Nash equilibria is neither strongly stable nor strongly unstable.

Proof. Let us denote the vectors of the canonical basis of $\mathbb{R}^{n_1+n_2}$ by

$$e_1[1l], \ldots, e_{l_1}[1l]\ ,\ e_1[1m], \ldots, e_{n_1-(l_1+k_1)}[1m]\ ,\ e_1[1r], \ldots, e_{k_1}[1r]$$

and

$$e_1[2l], \ldots, e_{l_2}[2l]\ ,\ e_1[2m], \ldots, e_{n_2-(l_2+k_2)}[2m]\ ,\ e_1[2r], \ldots, e_{k_2}[2r]\ .$$

Let $w_1 = \sum_{i=1}^{k_1} e_i[1r]$ and $w_2 = \sum_{i=1}^{k_2} e_i[2r]$. We obtain that

$$\begin{aligned}
DG(e_i[1l]) &= \lambda[1l]e_i[1l] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2\ ,\\
DG(e_i[1m]) &= \lambda[1m]e_i[1m] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2\ ,\\
DG(e_i[1r]) &= -p(1-p)A_{11}e_i[1r] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2\ ,\\
DG(e_i[2l]) &= \lambda[2l]e_i[1l] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2\ ,\\
DG(e_i[2m]) &= \lambda[2m]e_i[1m] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2\ ,\\
DG(e_i[2r]) &= -q(1-q)A_{22}e_i[1r] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2\ .
\end{aligned}$$

Furthermore,

$$\begin{aligned}
DG(w_1) &= \sum_{i=1}^{k_1} DG(e_i[1r]) = \sum_{i=1}^{k_1}\left(\sum_{j=1}^{k_1} p(1-p)A_{11}e_j[1r] + \sum_{j=1}^{k_2} q(1-q)A_{21}e_j[2r]\right)\\
&= (k_1-1)p(1-p)A_{11}w_1 + k_1 q(1-q)A_{21}w_2\ .
\end{aligned}$$

$$
\begin{aligned}
DG(w_2) &= \sum_{i=1}^{k_2} DG(e_i[2r]) = \sum_{i=1}^{k_2}\left(\sum_{j=1}^{k_2} q(1-q)A_{22}e_j[2r] + \sum_{j=1}^{k_1} p(1-p)A_{12}e_j[1r]\right) \\
&= k_2 p(1-p)A_{12}w_1 + (k_2-1)q(1-q)A_{22}w_2 \ .
\end{aligned}
$$

Hence, the space spaned by w_1 and w_2 is DG invariant and $DG| < w_1, w_2 >$ is given by the matrix $M(k_1, k_2)$. Therefore, the eigenvalues of $DG| < w_1, w_2 >$ are the eigenvalues of the matrix $M(k_1, k_2)$. For $t \in \{1, 2\}$ and $s \in \{l, m, r\}$, the vectors $e_i[ts] - e_j[ts]$, with $i \neq j$, belong to the eigenspace of the eigenvalue $\lambda[ts]$. Since the space spaned by w_1 and w_2 is DG invariant, the vectors $e_i[ts]$ belong to the extended eigenspace of the eigenvalue $\lambda[ts]$. □

In Figure 4, we observe the appearance of periodic cycles for the replicator dynamics. The individuals keep modifying their decisions along time exhibiting a periodic pattern in their decisions.

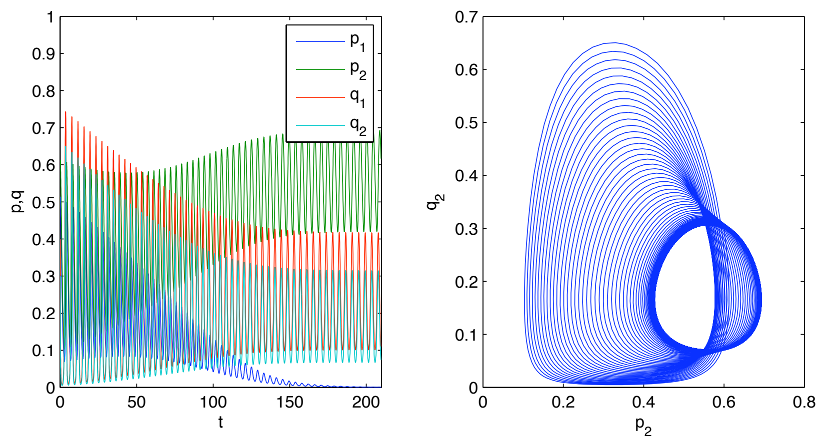

FIGURE 4. The time evolution of the probabilities of individuals under the replicator dynamics form a cycle. The influence matrix is given by $A_{11} = -0.1$, $A_{12} = 3$, $A_{21} = -10$, $A_{22} = 0$, $(n_1, n_2) = (2, 2)$ and $(x, y) = (0.4, 0.6)$.

7. Leadership Model.

A leader is an individual who can influence others to make a certain decision. For simplicity, we assume that the leader will influence other individuals to make decision Y. We study how the choice of the leader can influence the potential followers (individuals of type t_1 and t_2) to make the decision he pretends, see [3].

As in [3], the parameters (θ_i, P_i, L_i), with $i \in \{1, 2\}$, characterize the leaders and the potential followers. The types of leaders are defined as follows:

- *Altruistic, individualist and biased leaders.* The leader donates P_1 to the individuals of type t_1 and P_2 to the individuals of type t_2. The *altruistic leader*, for the individuals with type t_i, is the one who distributes a valuation to potential followers making the decision Y, i.e. $P_i > 0$; while the *individualist* *leader*, for the individuals with type t_i, is the one who distributes a devaluation or debt to potential followers making the decision Y, i.e. $P_i < 0$. The *biased leader* is the one who distributes a valuation to one type of potential followers and a debt to the other type of potential followers, i.e. $P_1 P_2 < 0$.

- *Consumption or wealth creation by the followers.* Define $\theta_1,\ \theta_2$ as the parameters of the consumption or wealth creation on the valuation distributed by the leader to other individuals. Therefore, the new valuation of the individuals, with type t_1 and t_2, to make decision Y is given by
$$\omega_i^Y + \frac{\theta_i P_i}{n_i},$$
where ω_i^Y corresponds to the valuation before the influence of the leader of the individuals to make decision Y. There is *wealth creation by the followers* of type t_i when $P_i > 0$ and $\theta_i > 1$ or when $P_i < 0$ and $\theta_i < 1$. There is *wealth consumption by the followers* of type t_i when $P_i > 0$ and $\theta_i < 1$ or when $P_i < 0$ and $\theta_i > 1$.

- *Influential and persuasive leader.* The influence or persuasiveness of the leader on other individuals is measured by the parameters L_1 and L_2. The individuals have a new valuation, when they make the decision N, under the influence of the leader, given by
$$\omega_i^N - L_i.$$
If $L_i < 0$, the individuals with type t_i will like more to make the decision that the leader pretends; however, if $L_i > 0$, the individuals with type t_i will like more to make the opposite decision from the one that the leader pretends.

Let
$$H = \max\{H(Y,Y), H(Y,N), H(N,Y), H(N,N)\}$$
and
$$V = \max\{V(Y,Y), V(Y,N), V(N,Y), V(N,N)\}.$$

Lemma 7.1. *Let S be a (mixed) Nash equilibrium.*

(i) If $\frac{\theta_1 P_1}{n_1} + L_1 > H - x$, then the individuals with type t_1 make the decision Y.

(ii) If $\frac{\theta_2 P_2}{n_2} + L_2 > V - y$, then the individuals with type t_2 make the decision Y.

As in [3], the inequalities above provide a sufficient condition in the values of the donated parts P_1 and P_2, in the values of the influence and persuasiveness L_1 and L_2 of the leader and in the values of the consumption or creation of wealth θ_1 and θ_2 by the followers, implying that the potential followers make the same decision as the leader.

8. **Conclusions.** The union of the equilibria $E(l_1, k_1, p; l_2, k_2, q)$ form hystereses. The equilibria $S(l_1; l_2)$ are the stable part of the hystereses and the equilibria $U(l_1, k_1, p; l_2, k_2, q)$ are the unstable part of the hystereses. The equilibria in $E(l_1, k_1, p; l_2, k_2, q) \setminus \{S(l_1, k_1, p; l_2, k_2, q), U(l_1, k_1, p; l_2, k_2, q)\}$ can be stable or unstable equilibria. Hence, small changes in the coordinates of the influence matrix that determines the equilibria sets $S(l_1; l_2)$ and $U(l_1, k_1, p; l_2, k_2, q)$ can create or annihilate cohesive and

disparate Nash equilibria giving rise to abrupt and collective changes in the decisions of the individuals that are explained by the hystereses. Furthermore, we observed the appearance of periodic attracting cycles and so the individuals can keep changing their decisions with a periodic pattern. We demonstrated how the characteristics of the leader can have a positive or negative influence over the decisions of the individuals. In particular, we show that an individualist leader might have to be more persuasive than an altruistic leader to convince the individuals to make a particular decision.

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