Asymmetric Combination of Logics is Functorial *

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Abstract. Asymmetric combination of logics is a formal process that develops the characteristic features of a specific logic on top of another one. Typical examples include the development of temporal, hybrid, and probabilistic dimensions over a given base logic. This paper argues that this sort of combination, at least in the scope of the examples here discussed, possesses a functorial nature. Such a view gives rise to several interesting questions. They range from the problem of combining translations (between logics), to that of ensuring property preservation along the process, and the way different asymmetric combinations can be related through appropriate natural transformations.

1 Introduction

1.1 Motivation and Context

It is well known that software's inherent high complexity renders formal design and analysis a difficult challenge, still largely unmet by the current engineering practices. Often, in fact, the formal specification of a non trivial software system calls for different logics in order to capture specific types of requirement, or design issues: if properties of data structures are typically captured in an equational framework, behavioural issues will call for some sort of modal or temporal logic, whereas probabilistic reasoning will be required in order to predict or analyse faulty behaviour in distributed systems.

This explains the growing interest in the systematic combination of logics, an area whose overall aim can be summed up in a simple methodological principle: *distinguish the underlying nature of the requirements to be formalised, and then build a single logic for the whole system by combining whatever logics are suitable to handle the types of requirement found.* Actually, this idea was already stressed

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in the eighties by J. Goguen and J. Meseguer, even though only more recently it started to gain prominence (*cf.* [3, 13]).

The current paper concerns *asymmetric* combination of logics, where the characteristic features of a logic are developed on top of another one (whenever found suitable we will drop the qualifier *asymmetric* or the subject *logics* in the expression). Probably the most famous example is the process of *temporalisation* [10], in which the features of a temporal logic are added on top of another logic, often referred to as the *base logic* to distinguish the original machinery from what was added. In brief, temporalisation adds a temporal dimension to the models of a given logic, as well as syntactical machinery to suitably handle the added dimension. The *hybridisation* [14] and *probabilisation* [2] processes are more recent examples. The former develops a *hybrid* logic [1] on top of the base one; the latter, adds probabilistic features instead. Other examples include *quantisation* [4] and *modalisation* [9], bringing into the picture features of quantum and modal logic, respectively.

Is there a common characterisation of these different combinations, able to provide a suitable setting to discuss their properties? Such is the question addressed in this paper.

Our approach is based on the theory of *institutions* [12], an abstract characterisation of logical system that encompasses syntax, semantics, and satisfaction. Put forward by J. Goguen and R. Burstall in the late seventies, its original aim was to develop as much Computing Science as possible in a general, uniform way, independently of any particular logical system. This has now been achieved to an extent even greater than originally thought. Indeed, institutions underlie the foundations of algebraic specification methods, and are most useful in handling and combining different sorts of logical systems to reason about computational phenomena. The universal character and resilience of institutions is witnessed by the wide set of logics formalised and subsequently explored within the framework. Examples go from the standard classical logics, to the most unconventional ones, typically capturing modern specification and programming paradigms. To be concrete, examples include *process algebras* [17], *temporal logics* [6], the Alloy language [18], coalgebraic logics [7], functional and imperative languages [24], among with many others.

1.2 Contributions and Roadmap

Institutions are objects of a well known category **I** (*cf.* [8, 15, 24]) whose arrows are translations between them. In this setting we argue that an asymmetric combination of logics can, very often, be seen as an *endofunctor* over **I**. Three examples (temporalisation, hybridisation, and probabilisation) are discussed in detail, with their definitions (slightly) reworked to fit in the general picture. Such a functorial perspective has several advantages: one is the possibility to lift the combination process from logics to their translations, which also allows the characterisation of natural transformations between asymmetric combinations. Another interesting possibility is the study of adjoints, and property preservation like conservativity, equivalence, and (co)limits.

As related work, the method for *parametrisation* of logics proposed by C. Caleiro *et al.* [5] should be mentioned. In brief, a logic is parametrised by another one if an atomic part of the first is replaced by the second; hence, the method distinguishes a parameter to fill (the atomic part), a parametrised logic (the 'top' logic) and a parameter logic (the logic inserted within). More recently, J. Rasga *et al.* [22, 23] proposed a method for importing logics by exploiting a graph-theoretic approach.

In Section 2 we recall the construction of the category of institutions that defines our framework, and slightly change the definitions of the three combinations. Then, in Section 3, they are enriched to become functorial. For the sake of simplicity and conciseness, we define an institutional abstract notion of asymmetric combination and make, to a large extent, the necessary proofs at this level of abstraction. We stress, however, that the paper's main objective is not to introduce such a notion, but rather to show that a number of asymmetric combinations possess a functorial nature; and that this perspective paves the way to several interesting mechanisms.

In the same section we study property preservation by these three (new) functors in what concerns conservativity (an important property that concerns the validation of specifications) and equivalence of institutions. We also explore the notion of natural transformation between asymmetric combinations. Finally, in Section 4, we conclude and suggest future lines of research.

This paper assumes that the reader has basic knowledge on Category Theory. Whenever found suitable, we will omit subscripts in natural transformations, and denote the underlying class of objects of any category $\mathbf{C}$ by $|\mathbf{C}|$ or just $\mathbf{C}$.

2 Preliminaries

2.1 Institutions

Let us recall the core notions of the theory of institutions and revisit the three working examples of combinations.

Definition 1. *An institution* $\mathcal{I}$ *is a tuple* $(Sign^{\mathcal{I}}, Sen^{\mathcal{I}}, Mod^{\mathcal{I}}, (\models^{\mathcal{I}}_{\Sigma})_{\Sigma \in |Sign^{\mathcal{I}}|})$ *where*

- $Sign^{\mathcal{I}}$ *is a category whose objects are signatures and arrows signature morphisms.*
- $Sen^{\mathcal{I}} : Sign^{\mathcal{I}} \to \mathbf{Set}$*, is a functor that for each signature* $\Sigma \in |Sign^{\mathcal{I}}|$ *returns a set of* Σ*–sentences,*
- $Mod^{\mathcal{I}} : (Sign^{\mathcal{I}})^{op} \to \mathbf{Cat}$*, is a functor that for each signature* $\Sigma \in |Sign^{\mathcal{I}}|$ *returns a category whose objects are* Σ*-models and arrows* Σ*–model homomorphisms.*
- $\models^{\mathcal{I}}_{\Sigma} \subseteq |Mod^{\mathcal{I}}(\Sigma)| \times Sen^{\mathcal{I}}(\Sigma)$*, is a satisfaction relation such that for each signature morphism* $\varphi : \Sigma \to \Sigma'$ *the following property holds*

$$Mod^{\mathcal{I}}(\varphi)(M) \models^{\mathcal{I}}_{\Sigma} \rho \text{ iff } M \models^{\mathcal{I}}_{\Sigma'} Sen^{\mathcal{I}}(\varphi)(\rho)$$

for any $M \in |Mod^{\mathcal{I}}(\Sigma')|$, $\rho \in Sen^{\mathcal{I}}(\Sigma)$.

Notation 1. *In the sequel we will refer to* $Mod^{\mathcal{I}}(\varphi)(M)$ *as the* φ*–reduct of* M *and denote it by* $M \upharpoonright_{\varphi}$. *When clear from the context, both the subscript and superscript in the satisfaction relation will be dropped.*

Definition 2. *Consider two institutions* $\mathcal{I}, \mathcal{I}'$. *A* comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ *is a triple such that*

- Φ: $Sign^{\mathcal{I}} \to Sign^{\mathcal{I}'}$ *is a functor,*
- α: $Sen^{\mathcal{I}} \to Sen^{\mathcal{I}'} \cdot \Phi$ *is a natural transformation,*
- β: $Mod^{\mathcal{I}'} \cdot \Phi^{op} \to Mod^{\mathcal{I}}$ *is a natural transformation* [3],
- *and for any* $\Sigma \in |Sign^{\mathcal{I}}|$, $M \in |Mod^{\mathcal{I}'} \cdot \Phi^{op}(\Sigma)|$ *and* $\rho \in Sen^{\mathcal{I}}(\Sigma)$

$$\beta_{\Sigma}(M) \models^{\mathcal{I}}_{\Sigma} \rho \text{ iff } M \models^{\mathcal{I}'}_{\Phi(\Sigma)} \alpha_{\Sigma}(\rho)$$

Definition 3. *Let us consider two comorphisms* $(\Phi_1, \alpha_1, \beta_1) : \mathcal{I} \to \mathcal{I}'$, *and* $(\Phi_2, \alpha_2, \beta_2) : \mathcal{I}' \to \mathcal{I}''$. *Their composition* $(\Phi_2, \alpha_2, \beta_2) \,;\, (\Phi_1, \alpha_1, \beta_1) : \mathcal{I} \to \mathcal{I}''$ *is defined as* $(\Phi_2, \alpha_2, \beta_2) \,;\, (\Phi_1, \alpha_1, \beta_1) \triangleq (\Phi_2 \cdot \Phi_1,\ (\alpha_2 \circ 1_{\Phi_1}) \cdot \alpha_1,\ \beta_1 \cdot (\beta_2 \circ 1_{\Phi_1^{op}}))$ *where the white circle denotes the* Godement (horizontal) *composition of natural transformations. Thus,*

$$\begin{aligned}
&\Phi_2 \cdot \Phi_1 \ :\ Sign^{\mathcal{I}} \to Sign^{\mathcal{I}''},\\
&(\alpha_2 \circ 1_{\Phi_1}) \cdot \alpha_1 \ :\ Sen^{\mathcal{I}} \to Sen^{\mathcal{I}''} \cdot \Phi_2 \cdot \Phi_1,\\
&\beta_1 \cdot (\beta_2 \circ 1_{\Phi_1^{op}}) \ :\ Mod^{\mathcal{I}''} \cdot \Phi_2^{op} \cdot \Phi_1^{op} \to Mod^{\mathcal{I}}.
\end{aligned}$$

Each institution $\mathcal{I}$ has as the identity comorphism the triple $(1_{Sign^{\mathcal{I}}}, 1_{Sen^{\mathcal{I}}}, 1_{Mod^{\mathcal{I}}})$.

Institutions and respective comorphisms form category **I**, mentioned in the introduction.

2.2 Asymmetric combinations of logics (institutionally)

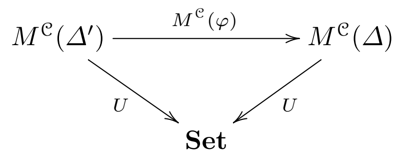
Considering separately each individual combination process leads to redundant proofs. For example, to show that all of them obey the functorial laws would require a separate proof for each case. For a more abstract treatment, we introduce the following characterisation of an asymmetric combination.
Let us start by considering categories $Sign_1$, $Sign_2$, and two functors

$$M^{\mathcal{C}} : (Sign_1)^{op} \to \mathbf{Cat},\ \ M^{\mathcal{I}} : (Sign_2)^{op} \to \mathbf{Cat}.$$

Assume that for each $\Delta \in |Sign_1|$ there is a functor $U_{(M^{\mathcal{C}}, \Delta)} : M^{\mathcal{C}}(\Delta) \to \mathbf{Set}$. Whenever no ambiguities arise, we will drop the subscript of $U_{(M^{\mathcal{C}}, \Delta)}$. Let us

[3] $(\_)^{op}$ applied to a functor $F : \mathbf{C} \to \mathbf{D}$ induces a functor $F^{op} : \mathbf{C}^{op} \to \mathbf{D}^{op}$ such that for any object or arrow a in $\mathbf{C}$, $F^{op}(a) = F(a)$.

further assume that given any morphism $\varphi : \Delta \to \Delta'$ of $Sign_1$, the induced functor $M^{\mathfrak{C}}(\varphi)$ makes the following diagram to commute.



This leads to functor $M^{\mathfrak{C}}(M^{\mathfrak{I}}) : (Sign_1 \times Sign_2)^{op} \to \mathbf{Cat}$ such that given any pair $(\Delta, \Sigma) \in Sign_1 \times Sign_2$, $M^{\mathfrak{C}}(M^{\mathfrak{I}})(\Delta, \Sigma)$ forms a *discrete* category whose objects are triples (S, R, m) where $R \in M^{\mathfrak{C}}(\Delta), U(R) = S$, and $m : S \to M^{\mathfrak{I}}(\Sigma)$. Moreover, given any signature morphism $\varphi_1 \times \varphi_2 : (\Sigma, \Delta) \to (\Sigma', \Delta')$ we have $M^{\mathfrak{C}}(M^{\mathfrak{I}})(\varphi_1 \times \varphi_2)\,(S, R, m) \triangleq (S,\ M^{\mathfrak{C}}(\varphi_1)(R),\ M^{\mathfrak{I}}(\varphi_2) \cdot m)$.

Definition 4. *An asymmetric combination* $\mathfrak{C}$ *is a tuple* $(Sign^{\mathfrak{C}}, Sen^{\mathfrak{C}}, M^{\mathfrak{C}}, \models^{\mathfrak{C}})$ *such that*

- $Sign^{\mathfrak{C}}$ *is a category of signatures.*
- $Sen^{\mathfrak{C}}$ *is a family of functions*

$$Sen^{\mathfrak{C}}_{Sign} : (Sign \to \mathbf{Set}) \to (Sign^{\mathfrak{C}} \times Sign \to \mathbf{Set})$$

  indexed by the categories Sign in **Cat**.
- $M^{\mathfrak{C}}$ *is a functor* $M^{\mathfrak{C}} : (Sign^{\mathfrak{C}})^{op} \to \mathbf{Cat}$ *as assumed above.*
- *Given functors* $M^{\mathfrak{I}} : Sign^{op} \to \mathbf{Cat}$, $Sen^{\mathfrak{I}} : Sign \to \mathbf{Set}$, $\models^{\mathfrak{C}}$ *is a family of relation liftings* $(\models^{\mathfrak{C}}_{(\Delta,\Sigma)})_{(\Delta,\Sigma) \in Sign^{\mathfrak{C}} \times Sign}$

$$\models^{\mathfrak{C}}_{(\Delta,\Sigma)} : |M^{\mathfrak{I}}(\Sigma)| \times Sen^{\mathfrak{I}}(\Sigma) \to |M^{\mathfrak{C}}(M^{\mathfrak{I}})\,(\Delta, \Sigma)| \times Sen^{\mathfrak{C}}(Sen^{\mathfrak{I}})(\Delta, \Sigma)$$

Given an institution $\mathfrak{I}$*, a pre institution* $\mathfrak{C}\mathfrak{I}$*, corresponding to a specific combination, is obtained as follows.*

- $Sign^{\mathfrak{C}\mathfrak{I}} \triangleq Sign^{\mathfrak{C}} \times Sign^{\mathfrak{I}}$.
- $Sen^{\mathfrak{C}\mathfrak{I}} \triangleq Sen^{\mathfrak{C}}(Sen^{\mathfrak{I}})$. *We will assume that the sentences given by* $Sen^{\mathfrak{C}\mathfrak{I}}$ *are inductively defined. Intuitively, their atoms will include the sentences of the base logic.*
- $Mod^{\mathfrak{C}\mathfrak{I}} \triangleq M^{\mathfrak{C}}(M^{\mathfrak{I}})$.
- *Given any signature* $(\Delta, \Sigma) \in |Sign^{\mathfrak{C}\mathfrak{I}}|$, $\models^{\mathfrak{C}\mathfrak{I}}_{(\Delta,\Sigma)} \triangleq \models^{\mathfrak{C}}_{(\Delta,\Sigma)} (\models^{\mathfrak{I}}_{\Sigma})$.

Temporalisation We are now ready to recast the three combinations of logics in the institutional setting. We start with temporalisation since it is the simplest of the three.

Definition 5. *Given an institution* $\mathfrak{I}$ *the temporalisation process returns pre institution* $\mathcal{L}\mathfrak{I} = (Sign^{\mathcal{L}\mathfrak{I}}, Sen^{\mathcal{L}\mathfrak{I}}, Mod^{\mathcal{L}\mathfrak{I}}, \models^{\mathcal{L}\mathfrak{I}})$ *defined as*

- Signatures. $Sign^{\mathcal{L}\mathfrak{I}} \triangleq Sign^{\mathcal{L}} \times Sign^{\mathfrak{I}}$*, where* $Sign^{\mathcal{L}}$ *is the one object category* 1. *Since* $Sign^{\mathcal{L}\mathfrak{I}} \cong Sign^{\mathfrak{I}}$*, no distinction will be made, unless stated otherwise, between the two signature categories.*

– SENTENCES. *Given a signature* $\Sigma \in |Sign^{\mathcal{L}\mathcal{I}}|$, $Sen^{\mathcal{L}\mathcal{I}}(\Sigma)$ *is the smallest set generated by grammar*

$$\rho \ni \; \psi \mid \neg\rho \mid \rho \wedge \rho \mid X\rho \mid \rho\, U \rho$$

where $\psi \in Sen^{\mathcal{I}}(\Sigma)$. *For any signature morphism* $\varphi : \Sigma \to \Sigma'$, $Sen^{\mathcal{L}\mathcal{I}}(\varphi)$ *is a function that, provided a sentence* $\rho \in Sen^{\mathcal{L}\mathcal{I}}(\Sigma)$, *replaces the base sentences* ψ (i.e. *elements of* $Sen^{\mathcal{I}}(\Sigma)$) *occurring in* ρ *by* $Sen^{\mathcal{I}}(\varphi)(\psi)$; *in symbols* $Sen^{\mathcal{L}\mathcal{I}}(\varphi)(\rho) = \rho[\psi \in Sen^{\mathcal{I}}(\Sigma) \,/\, Sen^{\mathcal{I}}(\varphi)(\psi)\,]$ (*recall that sentences are assumed to be inductively defined*).

– MODELS. *Given the object* $\star \in |1|$, $M^{\mathcal{L}}(\star)$ *is the category whose (unique) element is the pair* $(\mathbb{N}, \mathrm{suc} : \mathbb{N} \to \mathbb{N})$ ($\mathbb{N}$ *denotes the set of natural numbers*) *and* $U\,(\mathbb{N}, \mathrm{suc} : \mathbb{N} \to \mathbb{N})$ *is* $\mathbb{N}$. *Hence, the elements of category* $Mod^{\mathcal{L}\mathcal{I}}(\Sigma)$ *are triples* $(\mathbb{N}, \mathrm{suc} : \mathbb{N} \to \mathbb{N}, m)$ (*often denoted by letter* M) *where* $m : \mathbb{N} \to |Mod^{\mathcal{I}}(\Sigma)|$. *We will often denote* $m\,(n)$ *by* M_n.

– SATISFACTION. *Given a signature* $\Sigma \in |Sign^{\mathcal{L}\mathcal{I}}|$, $M \in |Mod^{\mathcal{L}\mathcal{I}}(\Sigma)|$, $\rho \in Sen^{\mathcal{L}\mathcal{I}}(\Sigma)$, $M \models \rho$ *iff* $M \models^0 \rho$ *where*

$M \models^j \psi$ *iff* $M_j \models \psi$ *for* $\psi \in Sen^{\mathcal{I}}(\Sigma)$
$M \models^j \rho \wedge \rho'$ *iff* $M \models^j \rho$ *and* $M \models^j \rho'$
$M \models^j \neg\rho$ *iff* $M \not\models^j \rho$
$M \models^j X\rho$ *iff* $M \models^{j+1} \rho$
$M \models^j \rho\, U\, \rho'$ *iff for some* $k \geq j$, $M \models^k \rho'$ *and for all* $j \leq i < k$, $M \models^i \rho$

Note that *temporalised* propositional logic coincides with the classic *linear temporal logic* (*cf.* [10]).

Theorem 1. *The satisfaction condition holds for* $\mathcal{L}\mathcal{I}$.

Proof. In appendix. □

Corollary 1. *Temporalised* $\mathcal{I}$ *(*i.e. $\mathcal{L}\mathcal{I}$*) is an institution.*

In the sequel we will see that the other two asymmetric combinations enjoy the same property, which is essential for their characterisation as endofunctors. Of course, this also entails the possibility of combining a logic an arbitrary number of times, using any of these three processes.

Probabilisation In order to handle probabilistic systems (*e.g. Markov chains*) probabilisation [2] adds a probabilistic dimension to logics. In institutional terms,

Definition 6. *Consider an arbitrary institution* $\mathcal{I}$. *Its probabilised version* $\mathcal{P}\mathcal{I} = (Sign^{\mathcal{P}\mathcal{I}}, Sen^{\mathcal{P}\mathcal{I}}, Mod^{\mathcal{P}\mathcal{I}}, \models^{\mathcal{P}\mathcal{I}})$ *is defined as follows*

– SIGNATURES. $Sign^{\mathcal{P}\mathcal{I}} \triangleq Sign^{\mathcal{P}} \times Sign^{\mathcal{I}}$, *where* $Sign^{\mathcal{P}}$ *is the one object category* 1. *Since* $Sign^{\mathcal{P}\mathcal{I}} \cong Sign^{\mathcal{I}}$, *no distinction will be made, unless stated otherwise, between the two signature categories.*

– SENTENCES. *For any* $\Sigma \in |Sign^{\mathcal{PI}}|$, $Sen^{\mathcal{PI}}(\Sigma)$ *is the smallest set generated by grammar*

$$\rho \ni \; t < t \mid \neg\rho \mid \rho \wedge \rho$$

for $t \in \mathrm{T}(\Sigma)$ $(\mathrm{T} : Sign^{\mathcal{PI}} \to \mathbf{Set})$. $\mathrm{T}(\Sigma)$ *is generated by grammar*

$$t \ni \; r \mid \textstyle\int \psi \mid t + t \mid t \,.\, t$$

where $r \in \mathbb{R}$ *(the set of real numbers) and* $\psi \in Sen^{\mathcal{I}}(\Sigma)$*. Also, we have*

$$Sen^{\mathcal{PI}}(\varphi)(\rho) \triangleq \rho[t \in \mathrm{T}(\Sigma) \,/\, \mathrm{T}(\varphi)(t)\,], \text{ where}$$
$$\mathrm{T}(\varphi)(t) \triangleq t[\,\psi \in Sen^{\mathcal{I}}(\Sigma) \,/\, Sen^{\mathcal{I}}(\varphi)(\psi)\,]$$

– MODELS. $Mod^{\mathcal{P}}(\star)$ *is the discrete category whose elements are probability spaces* $(S, p : 2^S \to [0,1])$*. Functor* U *returns the carrier set. Hence, models in* $Mod^{\mathcal{PI}}(\Sigma)$ *are triples* (S, p, m) *where* $m : S \to Mod^{\mathcal{I}}(\Sigma)$*. For each sentence* $\psi \in Sen^{\mathcal{I}}(\Sigma)$ *we have set* $m^{-1}[\psi] \triangleq \{s \in S : m(s) \models \psi\}$*.*
– SATISFACTION. *Finally, given a signature* $\Sigma \in |Sign^{\mathcal{PI}}|$*, a model* $M \in |Mod^{\mathcal{PI}}(\Sigma)|$*, and* $\rho \in Sen^{\mathcal{PI}}(\Sigma)$

$$\begin{array}{ll} M_r = r & \\ M_{(\int\psi)} = p(m^{-1}[\psi]) & M \models t < t' \text{ iff } M_t < M_{t'} \\ M_{(t+t')} = M_t + M_{t'} & M \models \neg\rho \quad \text{iff } M \not\models \rho \\ M_{(t.t')} = M_t \,.\, M_{t'} & M \models \rho \wedge \rho' \text{ iff } M \models \rho \text{ and } M \models \rho' \end{array}$$

Theorem 2. *The satisfaction condition holds for* $\mathcal{PI}$*.*

Proof. (a) The strictly less case is a direct consequence of Lemma 2 (in appendix).
(b) The negation and implication cases follow by induction on the structure of sentences.

□

Corollary 2. *Probabilised* $\mathcal{I}$ (i.e. $\mathcal{PI}$) *is an institution.*

Example 1. PROBABILISED PROPOSITIONAL LOGIC *(*$\mathcal{P}PL$*). The probabilisation of propositional logic returns the following logic:*

– SIGNATURES. *Signatures are sets of propositional symbols* P*.*
– SENTENCES. *Sentences are generated by grammar* $\rho \ni \; t < t \mid \neg\rho \mid \rho \wedge \rho$ *where* t *is a term generated by grammar* $t \ni \; r \mid \int\psi \mid t + t \mid t \,.\, t$ *for* $r \in \mathbb{R}$ *and* ψ *a propositional sentence.*
– MODELS. *Models are probability spaces equipped with a function whose domain is the set of outcomes and the codomain the universe of propositional models.*

With $\mathcal{P}$PL we are able to give a probabilistic 'flavour' to propositions, stating, for instance, that probability of p holding is less than probability of q holding, $\int p < \int q$. Other examples of probabilised logics are discussed in [2].

Hybridisation Hybridisation [14] provides the foundations for handling different kinds of *reconfigurable systems* (*i.e.*, computational systems that change their execution modes throughout their lifetime) in a systematic manner: in brief, the hybrid machinery relates and pinpoints the different execution modes while the base logic specifies the properties that are supposed to hold in each particular mode.

Since hybridisation was originally defined in institutional terms we will just repeat here (although with some minor differences) the original definition.

Definition 7. *Category* $Sign^{\mathcal{H}}$ *is the category* **Set** $\times$ **Set** *whose objects are pairs of sets* (Nom, Λ). *Nom denotes a set of nominal symbols, and* Λ *a set of modality symbols.*

Definition 8. *Given an institution* $\mathcal{I}$, $\mathcal{HI} = (Sign^{\mathcal{HI}}, Sen^{\mathcal{HI}}, Mod^{\mathcal{HI}}, \models^{\mathcal{HI}})$ *is defined as*

- SIGNATURES. $Sign^{\mathcal{HI}} \triangleq Sign^{\mathcal{H}} \times Sign^{\mathcal{I}}$.
- SENTENCES. *For any signature* $(\Delta, \Sigma) \in |Sign^{\mathcal{HI}}|$ *(with* $\Delta = (Nom, \Lambda)$*),* $Sen^{\mathcal{HI}}(\Delta, \Sigma)$ *is the smallest set generated by grammar*

$$\rho \ni \ i \mid \psi \mid \neg\rho \mid \rho \wedge \rho \mid @_i\rho \mid \langle\lambda\rangle\rho$$

  where $i \in Nom$, $\psi \in Sen^{\mathcal{I}}(\Sigma)$, $\lambda \in \Lambda$. *For any signature morphism* $\varphi_1 \times \varphi_2 : (\Delta, \Sigma) \to (\Delta', \Sigma')$, *nominals, modalities, and base sentences of* $\rho \in Sen^{\mathcal{HI}}(\Delta, \Sigma)$ *are replaced according to* $\varphi_1 \times \varphi_2$ *by* $Sen^{\mathcal{HI}}(\varphi_1 \times \varphi_2)$.
- MODELS. *Given a signature* $\Delta \in |Sign^{\mathcal{H}}|$, $M^{\mathcal{H}}(\Delta)$ *is the discrete category whose elements are triples* $(S, (R_i)_{i \in Nom}, (R_\lambda)_{\lambda \in \Lambda})$ *such that* $R_i \in S$, *and* $R_\lambda \subseteq S \times S$. *Functor U forgets the last two elements, keeping just the carrier set. For any signature morphism* $\varphi : (Nom, \Lambda) \to (Nom', \Lambda')$, *we have* $M^{\mathcal{H}}(\varphi)(S, (R'_i)_{i \in Nom'}, (R'_\lambda)_{\lambda \in \Lambda'}) \triangleq (S, (R_i)_{i \in Nom}, (R_\lambda)_{\lambda \in \Lambda})$, *where*

$$R_i = R'_{\pi_1(\varphi)(i)} \text{ and } R_\lambda = R'_{\pi_2(\varphi)(\lambda)}$$

- SATISFACTION. *Given* $(\Delta, \Sigma) \in |Sign^{\mathcal{HI}}|$, *a model* $M \in |Mod^{\mathcal{HI}}(\Delta, \Sigma)|$ *and a sentence* $\rho \in Sen^{\mathcal{HI}}(\Sigma)$, *the satisfaction relation is defined as*

$$M \models \rho \text{ iff } M \models^w \rho \text{ for all } w \in S$$

  where

  $M \models^w i$ *iff* $R_i = w$ *for* $i \in Nom$
  $M \models^w \psi$ *iff* $m(w) \models \psi$ *for* $\psi \in Sen^{\mathcal{I}}(\Sigma)$
  $M \models^w \neg\rho$ *iff* $M \not\models^w \rho$
  $M \models^w \rho \wedge \rho'$ *iff* $M \models^w \rho$ *and* $M \models^w \rho'$
  $M \models^w @_i\rho$ *iff* $M \models^{R_i} \rho$
  $M \models^w \langle\lambda\rangle\rho$ *iff there is some* $w' \in W$ *such that* $(w, w') \in R_\lambda$ *and* $M \models^{w'} \rho$

The proof that, for any institution $\mathcal{I}$, hybridisation yields another institution is given in references [14].

3 Asymmetric combinations of logics as functors

3.1 Lifting comorphisms

The three combinations of logics were revisited within the framework of institutions. We intend now to discuss them as translations between logics. We will do this at the level of the abstract definition of a combination of logics given above, leading thus to more powerful results, applicable not only to the three combinations discussed, but also to any other that fits the characterisation.

Formally, given a comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ we want any of the combination processes to map (Φ, α, β) into $\mathcal{C}(\Phi, \alpha, \beta) : \mathcal{C}\mathcal{I} \to \mathcal{C}\mathcal{I}'$. Actually, the strategy for such a lifting is simple: when transforming signatures, sentences or models, we keep the top level structure and change the bottom level according to the base comorphism. Thus,

Definition 9. *A comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ is lifted to mapping $(\mathcal{C}\Phi, \mathcal{C}\alpha, \mathcal{C}\beta) : \mathcal{C}\mathcal{I} \to \mathcal{C}\mathcal{I}'$ as follows:*

- Signatures. $\mathcal{C}\Phi : Sign^{\mathcal{C}\mathcal{I}} \to Sign^{\mathcal{C}\mathcal{I}'}$,

$$\mathcal{C}\Phi \triangleq 1_{Sign^{\mathcal{C}}} \times \Phi.$$

- Sentences. $\mathcal{C}\alpha : Sen^{\mathcal{C}\mathcal{I}} \to Sen^{\mathcal{C}\mathcal{I}'} \cdot \mathcal{C}\Phi$,

$$(\mathcal{C}\alpha)_{(\Delta,\Sigma)}(\rho) \triangleq \rho\,[\,\psi \in Sen^{\mathcal{I}}(\Sigma) \,/\, \alpha_{\Sigma}(\psi)\,],$$

  for any $(\Delta, \Sigma) \in |Sign^{\mathcal{C}\mathcal{I}}|$.
- Models. $\mathcal{C}\beta : Mod^{\mathcal{C}\mathcal{I}'} \cdot \mathcal{C}\Phi^{op} \to Mod^{\mathcal{C}\mathcal{I}}$,

$$(\mathcal{C}\beta)_{(\Delta,\Sigma)} \triangleq id \times id \times (\beta_{\Sigma}\cdot),$$

  for any $(\Delta, \Sigma) \in |Sign^{\mathcal{C}\mathcal{I}}|$.

Clearly, $\mathcal{C}\Phi$ is a functor, and $\mathcal{C}\alpha$, $\mathcal{C}\beta$ are natural transformations.

Lemma 1. *The lifting process, as defined above, preserves identities and distributes over composition.*

Proof. In appendix. □

To conclude that the three combinations are endofunctors one step remains: indeed we need to show that the lifted arrows are comorphisms. This, however, entails the need to inspect each specific combination on its own, as they all lift the satisfaction relation in different ways.

Theorem 3. *If (Φ, α, β) is a comorphism then, for any of three combinations $\mathcal{C}$ discussed above, $\mathcal{C}(\Phi, \alpha, \beta)$ is a comorphism as well.*

Proof. In appendix. □

3.2 Property preservation (conservativity and equivalence)

The characterisation of asymmetric combinations as endofunctors over the category of institutions **I** provides a sound basis for the study of property preservation by the corresponding combinations. Such a study is illustrated in this section where we prove that temporalisation, probabilisation, and hybridisation preserve conservativity and equivalence. We start with the former case.

In Computing Science a main reason to study under what conditions a logic may be translated into another is to seek for the existence of (better) computational proof support. In the institutional setting suitable translations are defined by comorphisms, which should additionally obey to the following condition: whenever completeness is required, *i.e.*, if one demands the validation of the specification against all possible scenarios (models), then the comorphisms involved must be conservative. Formally,

Definition 10. *A comorphism (Φ, α, β) is conservative whenever, for each signature $\Sigma \in |Sign^{\mathcal{I}}|$, β_Σ is surjective on objects.*

Let us describe in more detail the relevance of conservativity for validation. Recall the satisfaction condition placed upon comorphisms. For any $\Sigma \in |Sign^{\mathcal{I}}|$, $M \in |Mod^{\mathcal{I}'} \cdot \Phi^{op}(\Sigma)|$, and $\rho \in Sen^{\mathcal{I}}(\Sigma)$ we have $\beta_\Sigma(M) \models^{\mathcal{I}}_{\Sigma} \rho$ iff $M \models^{\mathcal{I}'}_{\Phi(\Sigma)} \alpha_\Sigma(\rho)$. Graphically, for each $\Sigma \in |Sign^{\mathcal{I}}|$

$$\begin{array}{ccc} Mod^{\mathcal{I}}(\Sigma) & \overset{\models^{\mathcal{I}}_{\Sigma}}{\rule[0.5ex]{5em}{0.4pt}} & Sen^{\mathcal{I}}(\Sigma) \\ {\scriptstyle \beta_\Sigma}\uparrow & & \downarrow{\scriptstyle \alpha_\Sigma} \\ Mod^{\mathcal{I}'} \cdot \Phi^{op}(\Sigma) & \underset{\models^{\mathcal{I}'}_{\Phi(\Sigma)}}{\rule[0.5ex]{5em}{0.4pt}} & Sen^{\mathcal{I}'} \cdot \Phi(\Sigma) \end{array}$$

Suppose we want to verify that a sentence $\rho \in Sen^{\mathcal{I}}(\Sigma)$ is satisfied by all models $M \in |Mod^{\mathcal{I}}(\Sigma)|$. For this we resort to the comorphism by translating the sentence (through α) into the target logic. The satisfaction condition tells us that if the sentence satisfies all models there, then all models in the image of β_Σ will satisfy the original sentence. Of course, if β_Σ is surjective on objects its image will coincide with $|Mod^{\mathcal{I}}(\Sigma)|$ and thus we prove that the original sentence satisfies all models in $|Mod^{\mathcal{I}}(\Sigma)|$.

Theorem 4. *A lifted conservative comorphism is still conservative.*

Proof. Consider a conservative comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$, we want to prove that for any signature $(\Delta, \Sigma) \in |Sign^{\mathcal{C}\mathrm{I}}|$ $(\mathcal{C}\beta)_{(\Delta,\Sigma)} = id \times id \times (\beta_\Sigma \cdot)$ is surjective on objects. Since identities are surjective we just need to show that each $f \in |Mod^{\mathcal{I}}(\Sigma)|^S$ has a function $g \in |Mod^{\mathcal{I}'} \cdot \Phi^{op}(\Sigma)|^S$ such that $f = \beta_\Sigma \cdot g$. Clearly, the condition for this to be true is that $img(f) \subseteq img(\beta_\Sigma)$, but the only way to ensure it is to have $img(\beta_\Sigma) = |Mod^{\mathcal{I}}(\Sigma)|$. In other words, β_Σ must be surjective on objects, which is given by the assumption.

$\square$

Next we show that the application of temporalisation, probabilisation, and hybridisation to two equivalent logics yields again two equivalent logics.

First, recall the definition of equivalence of categories.

Definition 11. *Two categories* $\mathbf{C}, \mathbf{D}$ *are said to be equivalent if there are two functors* $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ *and two natural isomorphisms* $\epsilon : FG \to 1_{\mathbf{D}}$, $\eta : 1_{\mathbf{C}} \to GF$.

We say that G (resp. F) is the inverse up to isomorphism of F (resp. G). Also, we call F an equivalence of categories.

Definition 12. *A comorphism* (Φ, α, β) *is an institution equivalence if the following conditions hold.*

- Signatures. *Φ forms an equivalence of categories.*
- Sentences. *α has an inverse up to semantical equivalence,* i.e.*, a natural transformation* $\alpha^{-1} : Sen^{\mathcal{I}'} \cdot \Phi \to Sen^{\mathcal{I}}$ *such that for any sentence* $\rho \in Sen^{\mathcal{I}}(\Sigma)$,
$$(\alpha^{-1} \cdot \alpha)(\rho) \models \rho, \quad \rho \models (\alpha^{-1} \cdot \alpha)(\rho)$$
or more concisely, $(\alpha^{-1} \cdot \alpha)(\rho) \models\mid \rho$.
Moreover, for any sentence $\rho \in Sen^{\mathcal{I}'} \cdot \Phi(\Sigma)$, $(\alpha \cdot \alpha^{-1})(\rho) \models\mid \rho$.
- Models. *β has an inverse up to isomorphism,* i.e.*, a natural transformation* β^{-1} *such that for any* $\Sigma \in |Sign^{\mathcal{I}}|$, *functor* β^{-1}_{Σ} *is the inverse up to ismorphism of* β_{Σ}.

More about equivalence of institutions can be found in documents [8, 15].

Theorem 5. *A lifted institution equivalence is still an institution equivalence.*

Proof. Suppose that (Φ, α, β) is an institution equivalence. Then,

- Signatures. Since Φ is an equivalence of categories, $\mathcal{C}\Phi = 1_{Sign^{\mathcal{C}}} \times \Phi$ must be as well.
- Sentences. To show that for any $\rho \in Sen^{\mathcal{C}\mathcal{I}}(\Delta, \Sigma)$, property $\big((\mathcal{C}\alpha)^{-1} \cdot \mathcal{C}\alpha\big)(\rho) \models\mid \rho$ holds is, by definition of $\mathcal{C}\alpha$, equivalent to showing that
$$\rho[\psi \in Sen^{\mathcal{I}}(\Sigma) \,/\, (\alpha^{-1} \cdot \alpha)(\psi)\,] \models\mid \rho$$
This boils down to proving that $(\alpha^{-1} \cdot \alpha)(\psi) \models\mid \psi$ for any $\psi \in Sign^{\mathcal{I}}(\Sigma)$ which is given by the assumption.
The proof that $\big(\mathcal{C}\alpha \cdot (\mathcal{C}\alpha)^{-1}\big)(\rho) \models\mid \rho$ is analogous.
- Models. Finally, we need to show that for any $(\Delta, \Sigma) \in |Sign^{\mathcal{C}I}|$, $(\mathcal{C}\beta)_{(\Delta,\Sigma)}$ has an inverse up to isomorphism. For this we lift β^{-1}_{Σ} (given by the assumption) into $(\mathcal{C}\beta)^{-1}_{(\Delta,\Sigma)} = (id \times id \times \beta^{-1}_{\Sigma} \cdot)$. Since β^{-1}_{Σ} is an inverse up to isomorphism of β_{Σ} it is clear that $(\mathcal{C}\beta)^{-1}_{(\Delta,\Sigma)}$ is also an inverse up to isomorphism of $(\mathcal{C}\beta)_{(\Delta,\Sigma)}$.

□

3.3 Natural transformations

We consider now natural transformations between asymmetric combinations of logics, which seem to fit nicely into the picture: while lifted comorphisms map the bottom level and keep the top, such natural transformations map the top and keep the bottom. For example, take a natural transformation $\tau : \mathcal{L} \to \mathcal{H}$. It is clear that each institution $\mathcal{I}$, induces a comorphism $\tau_{\mathcal{I}} : \mathcal{L}\mathcal{I} \to \mathcal{H}\mathcal{I}$. Furthermore, naturality expresses commutativity of the diagram below

$$\begin{array}{ccc} \mathcal{L}\mathcal{I} & \xrightarrow{\mathcal{L}(\Phi,\alpha,\beta)} & \mathcal{L}\mathcal{I}' \\ {\scriptstyle \tau_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathcal{I}'}} \\ \mathcal{H}\mathcal{I} & \xrightarrow[\mathcal{H}(\Phi,\alpha,\beta)]{} & \mathcal{H}\mathcal{I}' \end{array}$$

for each comorphism (Φ, α, β). This means that when translating a logic whose both levels are mapped by a composition of natural transformations and lifted comorphisms, it does not matter which of the top or bottom level is taken first.

Let us illustrate this construction through natural transformation $\tau : \mathcal{L} \to \mathcal{H}$, which relates temporalisation to hybridisation. We will, for now, disregard the *until* (U) constructor associated with $\mathcal{L}$. First consider a signature $N \in |Sign^{\mathcal{H}}|$ such that $N \triangleq (\{Init\}, \{After, After^{\star}, Next\})$. Then for any signature $(N, \Sigma) \in |Sign^{\mathcal{H}I}|$ define the subcategory of $Mod^{\mathcal{H}\mathcal{I}}(N, \Sigma)$ (denoted in the sequel by $M^{\mathcal{N}\mathcal{I}}(N, \Sigma)$) whose objects are triples (S, R, m) that obey to the following rules:

$S = \mathbb{N}$	
$R_{Init} = 0$	$(a, b) \in R_{After}$ iff $a < b$
$(a, b) \in R_{Next}$ iff $b = \mathrm{suc}(a)$	$(a, b) \in R_{After^{\star}}$ iff $a \leq b$.

Definition 13. *Given an institution* $\mathcal{I}$*, define an arrow* $\tau_{\mathcal{I}} = (\tau_{\mathcal{I}}\Phi, \tau_{\mathcal{I}}\alpha, \tau_{\mathcal{I}}\beta)$ (*whose subscripts we will omit whenever no ambiguities arise*) *where*

– SIGNATURES. $\tau\Phi : Sign^{\mathcal{L}\mathcal{I}} \to Sign^{\mathcal{H}\mathcal{I}}$ *is a functor such that* $\tau\Phi\,(\Sigma) \triangleq (Seq, \Sigma)$ *and, for any signature morphism* $\varphi : \Sigma \to \Sigma'$

$$\tau\Phi\,(\varphi) : (Seq,\ \Sigma) \to (Seq,\ \Sigma'), \quad \tau\Phi\,(\varphi) \triangleq id \times \varphi$$

– SENTENCES. *Given any signature* $\Sigma \in |Sign^{\mathcal{L}\mathcal{I}}|$, $\tau\alpha : Sen^{\mathcal{L}\mathcal{I}}(\Sigma) \to Sen^{\mathcal{H}\mathcal{I}} \cdot \tau\Phi(\Sigma)$ *is a function such that* $\tau\alpha(\rho) \triangleq @_{Init}\sigma(\rho)$ *where*

$\sigma(\psi) \;= \psi,$ *for* $\psi \in Sen^{\mathcal{I}}(\Sigma)$	$\sigma(\rho \wedge \rho') = \sigma(\rho) \wedge \sigma(\rho')$
$\sigma(\neg\rho) = \neg\sigma(\rho)$	$\sigma(X\rho) \;= [Next]\ \sigma(\rho)$

The proof that $\tau\alpha$ *is a natural transformation follows through routine calculation.*

– *Finally, given any signature* $\Sigma \in |Sign^{\mathcal{L}\mathcal{I}}|$*, arrow* $\tau\beta : Mod^{\mathcal{H}\mathcal{I}} \cdot (\tau\Phi)^{op} \to Mod^{\mathcal{L}\mathcal{I}}$ *is a functor such that*

$$\tau\beta\,(S, R, m) \triangleq (\mathbb{N}, \mathrm{suc} : \mathbb{N} \to \mathbb{N}, m)$$

Clearly, $\tau\beta$ is a natural transformation.

Theorem 6. *$\tau : \mathcal{L} \to \mathcal{H}$ forms a natural transformation whenever $Mod^{\mathcal{H}\mathcal{I}}$ (for any institution $\mathcal{I}$) is equal to $Mod^{\mathcal{N}\mathcal{I}}$.*

Proof. In appendix. □

In order to include the *until* constructor we need to add *nominal quantification* to hybridisation, which would yield translation

$$\sigma(\rho U \rho') = \exists x\,.\,\langle After^{\star}\rangle(\,x \wedge \sigma(\rho')) \wedge [After^{\star}](\langle After\rangle x \Rightarrow \sigma(\rho))$$

Actually, to show that hybridisation with nominal quantification is also an endofunctor (and the satisfaction condition for *until* associated with τ holds) boils down to a routine calculation. This means that the theorem above can be replicated, but with the *until* operator, in a straightforward manner.

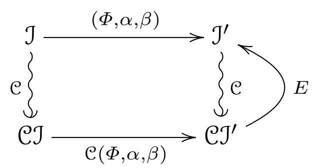
4 Conclusions and future work

Asymmetric combination of logics is a promising tool for the (formal) development of complex, heterogenous software systems. This justifies their study at an abstract level, paving the way to general results on, for example, property preservation along the combination process. Often such a study has been made on a case-by-case basis (as in, for example, [9] or [20, 21]). Here we adopted a more general perspective, closer to the work of C. Caleiro *et al.* [5] and J. Rasga *et al.* [22, 23], but from an institutional point of view, which naturally leads to the question *what are asymmetric combination processes within the category of institutions?*

This paper provided their characterisation as endofunctors over the category of institutions by showing how to lift comorphisms and proving that the lifted arrows obey the functorial laws. This made clear that not only logics, but also their translations can be combined. In the process we developed an institutional, abstract notion of asymmetric combination of logics.

Moreover, this work hints at a set of research directions that we have only grasped in this paper. For example, we proved, at the abstract level, that conservativity (a fundamental property for safely 'borrowing' a theorem prover) and equivalence are preserved by combination, but a full study needs to be done in what regards preservation of (co)limits, namely to discuss whether *the combination of the product of two logics is equivalent to the product of the corresponding combination of the same logics.*

Another research direction was set by J. Goguen in his Categorial Manifest [11]: *"if you have found an interesting functor, you might be well advised to investigate its adjoints"*. We studied natural transformations between such functors and showed that they nicely complement the lifting of comorphisms: while the latter map the bottom level and keep the top, the former map the top and keep the bottom. We gave an example of a natural transformation between temporalisation and hybridisation, but others deserve to be studied as well. For example, in [14] the authors show how, given a comorphism from an institution $\mathcal{I}$ to FOL, a comorphism from $\mathcal{HI}$ to FOL can be obtained. More generally, the current paper shows that comorphisms can be built by lifting the original comorphism and then composing it with the 'flat' natural transformation $E : \mathcal{C} \to 1_{\mathcal{I}}$ (if it exists). In diagrammatic form,



On a more theoretical note, the perspective taken in this paper also suggests to look at 'trivial' asymmetric combinations. For example, it is straightforward to define *identisation*, in which the added layer has a trivial structure, but also *trivialisation* ($\mathcal{T}$), which turns a logic into the trivial one (technically, the initial object in the category of institutions **I**). The latter case implies that there is a (unique) natural transformation $\mathcal{T} \to \mathcal{C}$ for any combination $\mathcal{C}$. Actually, one can even go further and show that $\mathcal{T}$ is the initial object in the category of endofunctors over **I**.

From a pragmatic point of view, the incorporation of these ideas in the HETS platform [16] paves the way for its effective use in the Software Engineering community. HETS is often described as a "motherboard" of logics where different "expansion cards" can be plugged in. These are individual logics (with their particular analysers and proof tools) as well as logic translations. To make them compatible, logics are formalised as institutions and translations as comorphisms. Therefore HETS provides an interesting setting for the implementation of the theory developed in this paper. Again, a specific case — that of *hybridisation* — has already been implemented in the HETS platform [19].

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Appendix (Proofs)

Lemma 2. *For any signature morphism $\varphi : \Sigma \to \Sigma'$, any model $M \in |Mod^{\mathcal{PI}}(\Sigma')|$, and any term $t \in T(\Sigma)$, $(M{\restriction_\varphi})_t = M_{T(\varphi)(t)}$*

Proof. By induction on the structure of terms,

(a)

$$
\begin{array}{ll}
 & (M{\restriction_{\varphi}})_r \\
= & \quad \{ \text{ interpretation of terms } \} \\
 & r \\
= & \quad \{ \text{ definition of } T(\varphi) \} \\
 & M_{T(\varphi)(r)}
\end{array}
$$

(b)

$$
\begin{array}{ll}
 & (M{\restriction_{\varphi}})_{(\int \psi)} \\
= & \quad \{ \text{ interpretation of terms } \} \\
 & p\,(\,(Mod^{\mathcal{I}}(\varphi)\,\cdot\,m)^{-1}[\psi]\,) \\
= & \quad \{ \text{ definition of } m^{-1}[\psi] \} \\
 & p\,(\,\{s \in S : Mod^{\mathcal{I}}(\varphi) \cdot m(s) \models \psi\}\,) \\
= & \quad \{ \ \mathcal{I} \text{ is an institution } \} \\
 & p\,(\,\{s \in S : m(s) \models Sen^{\mathcal{I}}(\varphi)(\psi)\}\,) \\
= & \quad \{ \text{ definition of } m^{-1}[\psi] \ \} \\
 & p\,(\,m^{-1}[\,Sen^{\mathcal{I}}(\varphi)(\psi)\,]\,) \\
= & \quad \{ \text{ interpretation of terms } \} \\
 & M_{\int Sen^{\mathcal{I}}(\varphi)(\psi)} \\
= & \quad \{ \text{ definition of } T(\varphi) \ \} \\
 & M_{T(\varphi)(\int \psi)}
\end{array}
$$

(c) All other cases are straightforward.

□

Proof of Theorem 1. By induction on the structure of sentences, namely for any $\psi \in Sen^{\mathcal{I}}(\Sigma)$

$$(M\restriction_\varphi) \models^j \psi$$
$$\Leftrightarrow \quad \{ \text{ definition of } \models^{\mathcal{L}\mathcal{I}} \}$$
$$(M\restriction_\varphi)_j \models \psi$$
$$\Leftrightarrow \quad \{ \text{ (reduct) definition of } Mod^{\mathcal{L}\mathcal{I}} \}$$
$$M_j\restriction_\varphi \models \psi$$
$$\Leftrightarrow \quad \{ \ \mathcal{I} \text{ is an institution } \}$$
$$M_j \models Sen^{\mathcal{I}}(\varphi)(\psi)$$
$$\Leftrightarrow \quad \{ \text{ definition of } Sen^{\mathcal{L}\mathcal{I}}(\varphi)\text{, definition of } \models^{\mathcal{L}\mathcal{I}} \}$$
$$M \models^j Sen^{\mathcal{L}\mathcal{I}}(\varphi)(\psi)$$

All other cases are straightforward.

Proof of Lemma 1. We start with preservation of identities.

(a) SIGNATURES.

$$\mathcal{C}(1_{Sign^{\mathcal{I}}})$$
$$= \quad \{ \text{ definition of } \mathcal{C}\Phi \}$$
$$1_{Sign^{\mathcal{C}}} \times 1_{Sign^{\mathcal{I}}}$$
$$= \quad \{ \ Sign^{\mathcal{C}} \times Sign^{\mathcal{I}} = Sign^{\mathcal{C}\mathcal{I}} \}$$
$$1_{Sign^{\mathcal{C}\mathcal{I}}}$$

(b) SENTENCES.

$$\mathcal{C}(1_{Sen^{\mathcal{I}}})_{(\Delta,\Sigma)}(\rho)$$
$$= \quad \{ \text{ definition of } \mathcal{C}\alpha \}$$
$$\rho[\, \psi \in Sen^{\mathcal{I}}(\Sigma) \,/\, (1_{Sen^{\mathcal{I}}})_\Sigma(\psi) \,]$$
$$= \quad \{ \text{ definition of } 1_{Sen^{\mathcal{I}}} \}$$
$$\rho$$

(c) MODELS.

$$\mathcal{C}(1_{Mod^{\mathcal{I}}})_{(\Delta,\Sigma)}$$
$$= \quad \{ \text{ definition of } \mathcal{C}\beta \}$$
$$id \times id \times \big((1_{Mod^{\mathcal{I}}})_\Sigma \cdot \big)$$
$$= \quad \{ \ id \cdot m = m \ \}$$
$$id \times id \times id$$

In the case of distribution over composition, we reason

(a) SIGNATURES. $\mathcal{C}(\Phi_2 \cdot \Phi_1) = \mathcal{C}\Phi_2 \cdot \mathcal{C}\Phi_1$

$$
\begin{aligned}
& \mathcal{C}(\Phi_2 \cdot \Phi_1) \\
= & \quad \{ \text{ definition of } \mathcal{C}\Phi \ \} \\
& 1_{Sign^{\mathcal{C}}} \times (\Phi_2 \cdot \Phi_1) \\
= & \quad \{ \text{ identity, and definition of product } \} \\
& (1_{Sign^{\mathcal{C}}} \times \Phi_2) \cdot (1_{Sign^{\mathcal{C}}} \times \Phi_1) \\
= & \quad \{ \text{ definition of } \mathcal{C}\Phi \text{ (twice) } \} \\
& \mathcal{C}\Phi_2 \cdot \mathcal{C}\Phi_1
\end{aligned}
$$

(b) SENTENCES. $\mathcal{C}\big((\alpha_2 \circ 1_{\Phi_1}) \cdot \alpha_1\big) = (\mathcal{C}\alpha_2 \circ 1_{\mathcal{C}\Phi_1}) \cdot \mathcal{C}\alpha_1$

$$
\begin{aligned}
& \mathcal{C}((\alpha_2 \circ 1_{\Phi_1}) \cdot \alpha_1)\ (\rho) \\
= & \quad \{ \text{ definition of } \mathcal{C}\alpha \text{, and composition of natural transformations } \} \\
& \rho[\psi \in Sen^{\mathcal{I}}(\Sigma)\ /\ (\alpha_2 \circ 1_{\Phi_1}) \cdot \alpha_1\ (\psi)] \\
= & \quad \{ \text{ horizontal composition } \} \\
& \rho[\psi \in Sen^{\mathcal{I}}(\Sigma)\ /\ \alpha_2 \cdot \alpha_1\ (\psi)] \\
= & \quad \{ \text{ composition } \} \\
& \big(\rho[\ \psi \in Sen^{\mathcal{I}}(\Sigma)\ /\ \alpha_1\ (\psi)\ ]\ \big)\ \ [\ \psi \in Sen^{\mathcal{I}'} \cdot \Phi_1(\Sigma)\ /\ \alpha_2\ (\psi)\ ] \\
= & \quad \{ \text{ horizontal composition } \} \\
& \big((\mathcal{C}\alpha_2) \circ 1_{\mathcal{C}\Phi_1}\big) \cdot (\mathcal{C}\alpha_1)(\rho) \\
= & \quad \{ \text{ composition of natural transformations } \} \\
& \big((\mathcal{C}\alpha_2 \circ 1_{\mathcal{C}\Phi_1}) \cdot \mathcal{C}\alpha_1\big)(\rho)
\end{aligned}
$$

(c) MODELS. $\mathcal{C}\big(\beta_1 \cdot (\beta_2 \circ 1_{\Phi_1^{op}})\big) = \mathcal{C}\beta_1 \cdot (\mathcal{C}\beta_2 \circ 1_{\mathcal{C}\Phi_1^{op}})$

$$
\begin{aligned}
& \mathcal{C}\big(\beta_1 \cdot (\beta_2 \circ 1_{\Phi_1^{op}})\big)_{(\Delta,\Sigma)} \\
= & \quad \{ \text{ definition of } \mathcal{C}\beta \ \} \\
& id \times id \times \big((\beta_1 \cdot (\beta_2 \circ 1_{\Phi_1^{op}})_{\Sigma} \cdot \big) \\
= & \quad \{ \text{ identity, and definition of product } \} \\
& \big(id \times id \times (\beta_1)_{\Sigma} \cdot \big) \cdot \big(id \times id \times (\beta_2 \circ 1_{\Phi_1^{op}})_{\Sigma} \cdot \big)
\end{aligned}
$$

$$
\begin{aligned}
=\quad & \{\text{ horizontal composition }\}\\
& \big(id \times id \times (\beta_1)_\Sigma \cdot\big) \cdot \big(id \times id \times (\beta_2)_{\Phi_1^{op}(\Sigma)} \cdot\big)\\
=\quad & \{\text{ definition of } \mathcal{C}\beta \text{ (twice) }\}\\
& (\mathcal{C}\beta_1)_{(\Delta,\Sigma)} \cdot (\mathcal{C}\beta_2)_{(\Delta,\Phi_1^{op}(\Sigma))}\\
=\quad & \{\text{ horizontal composition }\}\\
& (\mathcal{C}\beta_1)_{(\Delta,\Sigma)} \cdot (\mathcal{C}\beta_2 \circ 1_{\mathcal{C}\Phi_1^{op}})_{(\Delta,\Sigma)}\\
=\quad & \{\text{ composition of natural transformations }\}\\
& \big(\, \mathcal{C}\beta_1 \cdot (\mathcal{C}\beta_2 \circ 1_{\mathcal{C}\Phi_1^{op}})\,\big)_{(\Delta,\Sigma)}
\end{aligned}
$$

Proof of Theorem 3. We start with the case of temporalisation, which follows by induction on the structure of sentences.

(a) $\psi \in Sen^{\mathcal{I}}(\Sigma)$,

$$
\begin{aligned}
& (\mathcal{L}\beta)(M) \models^j \psi\\
\Leftrightarrow\quad & \{\text{ definition } \models^{\mathcal{L}\mathcal{I}} \}\\
& (\mathcal{L}\beta)(M)_j \models \psi\\
\Leftrightarrow\quad & \{\text{ definition of } \mathcal{L}\beta \}\\
& \beta(M_j) \models \psi\\
\Leftrightarrow\quad & \{\ (\Phi, \alpha, \beta) \text{ is a comorphism } \}\\
& M_j \models \alpha(\psi)\\
\Leftrightarrow\quad & \{\text{ definition of } \mathcal{L}\alpha \}\\
& M_j \models (\mathcal{L}\alpha)(\psi)\\
\Leftrightarrow\quad & \{\text{ definition of } \models^{\mathcal{L}\mathcal{I}} \}\\
& M \models^j (\mathcal{L}\alpha)(\psi)
\end{aligned}
$$

(b) $\neg\rho$,

$$
\begin{aligned}
& (\mathcal{L}\beta)(M) \models^j \neg\rho\\
\Leftrightarrow\quad & \{\text{ definition } \models^{\mathcal{L}\mathcal{I}} \}\\
& (\mathcal{L}\beta)(M) \not\models^j \rho\\
\Leftrightarrow\quad & \{\text{ induction hypothesis }\}\\
& M \not\models^j (\mathcal{L}\alpha)(\rho)\\
\Leftrightarrow\quad & \{\text{ definition of } \models^{\mathcal{L}\mathcal{I}} \text{ and } \mathcal{L}\alpha \}\\
& M \models^j (\mathcal{L}\alpha)(\neg\rho)
\end{aligned}
$$

(c) The remaining cases are analogous.

In the case of probabilistation we need the lemma below.

Lemma 3. *Consider a signature* $\Sigma \in |Sign^{\mathcal{PI}}|$*, a term* $t \in \mathrm{T}(\Sigma)$*, and a model* $M \in |Mod^{\mathcal{PI}'} \cdot \mathcal{P}\Phi^{op}(\Sigma)|$*. The following property holds.*

$$\big((\mathcal{P}\beta)\,(M)\big)_t = M_{(\mathcal{P}\alpha)(t)}$$

Proof. Follows by induction on the structure of terms.

(a)

$$\begin{aligned}
& \big((\mathcal{P}\beta)(M)\big)_r \\
= & \quad \{ \text{ interpretation of terms } \} \\
& M_r \\
= & \quad \{ \text{ definition of } \mathcal{P}\alpha \ \} \\
& M_{(\mathcal{P}\alpha)(r)}
\end{aligned}$$

(b)

$$\begin{aligned}
& \big(\,(\mathcal{P}\beta)(M)\,\big)_{\int\psi} \\
= & \quad \{ \text{ definition of } \mathcal{P}\beta \text{ interpretation of terms } \} \\
& p\,\big(\,(\beta\,\cdot\,m)^{-1}[\psi]\,\big) \\
= & \quad \{ \text{ definition of } m^{-1}[\psi] \ \} \\
& p\,\big(\,\{s \in S : \beta\,\cdot\,m(s) \models \psi\}\,\big) \\
= & \quad \{ \ (\Phi, \alpha, \beta) \text{ is a comorphism } \} \\
& p\,\big(\,\{\, s \in S : m(s) \models \alpha(\psi)\,\}\,\big) \\
= & \quad \{ \text{ definition of } m^{-1}[\psi] \ \} \\
& p\,\big(\,m^{-1}[\alpha(\psi)]\,\big) \\
= & \quad \{ \text{ definition of } \mathcal{P}\beta \text{ and interpretation of terms } \} \\
& M_{\int\alpha(\psi)} \\
= & \quad \{ \text{ definition of } \mathcal{P}\alpha \ \} \\
& M_{(\mathcal{P}\alpha)(\int\psi)}
\end{aligned}$$

(c) The sum and multiplication cases are proved in a similar way.

□

The satisfaction condition for $\mathcal{P}(\Phi, \alpha, \beta)$ follows by induction on the structure of sentences. In particular, the stricly less case is a direct consequence of the previous lemma. Negation and implication are proved as usual.

The case of hybridisation follows, again, by induction on the structure of sentences. Namely,

(a) $i \in Nom$,

$$\begin{aligned}
& \mathcal{H}\beta\,(M) \models^w i \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}} \} \\
& \big(\,\mathcal{H}\beta\,(M)\,\big)_i = w \\
\Leftrightarrow \quad & \{ \text{ definition of } \mathcal{H}\beta \} \\
& M_i = w \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}}, \text{ and } \mathcal{H}\alpha \} \\
& M \models^w \mathcal{H}\alpha\,(i)
\end{aligned}$$

(b) $\psi \in Sen^{\mathcal{I}}(\Sigma)$,

$$\begin{aligned}
& \mathcal{H}\beta\,(M) \models^w \psi \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}} \} \\
& \beta \cdot m(w) \models \psi \\
\Leftrightarrow \quad & \{ \text{ } (\Phi, \alpha, \beta) \text{ is an institution comorphism } \} \\
& m(w) \models \alpha\,(\psi) \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}}, \text{ and } \mathcal{H}\alpha \} \\
& M \models^w \mathcal{H}\alpha\,(\psi)
\end{aligned}$$

(c) $@_i\rho$,

$$\begin{aligned}
& \mathcal{H}\beta\,(M) \models^w @_i\rho \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}}, \text{ and } \big(\,\mathcal{H}\beta\,(M)\,\big)_i = M_i \} \\
& \mathcal{H}\beta\,(M) \models^{M_i} \rho \\
\Leftrightarrow \quad & \{ \text{ induction hypothesis } \} \\
& M \models^{M_i} \mathcal{H}\alpha\,(\rho) \\
\Leftrightarrow \quad & \{ \text{ definition of } \models^{\mathcal{H}} \} \\
& M \models^w @_i\,\mathcal{H}\alpha\,(\rho) \\
\Leftrightarrow \quad & \{ \text{ definition of } \mathcal{H}\alpha \} \\
& M \models^w \mathcal{H}\alpha\,(@_i\rho)
\end{aligned}$$

(d) $\langle\lambda\rangle\rho$,

$$\mathcal{H}\beta\,(M) \models^{w} \langle\lambda\rangle\rho$$

$\Leftrightarrow$ $\{$ definition of $\models^{\mathcal{H}}$, and R_λ of $\mathcal{H}\beta\,(M)$ is equal to R_λ of M $\}$

there is a w' such that $(w, w') \in R_\lambda$ and $\mathcal{H}\beta\,(M) \models^{w'} \rho$

$\Leftrightarrow$ $\{$ induction hypothesis $\}$

there is a w' such that $(w, w') \in R_\lambda$ and $M \models^{w'} \mathcal{H}\alpha\,(\rho)$

$\Leftrightarrow$ $\{$ definition of $\models^{\mathcal{H}}$ $\}$

$$M \models^{w} \langle\lambda\rangle(\mathcal{H}\alpha\,(\rho))$$

$\Leftrightarrow$ $\{$ definition of $\mathcal{H}\alpha$ $\}$

$$M \models^{w} \mathcal{H}\alpha\,(\langle\lambda\rangle\rho)$$

(e) The remaining cases are routine induction proofs.

Proof of Theorem 6. Follows by induction on the structure of sentences, in particular

(a) $\psi \in Sen^{\mathcal{I}}(\Sigma)$,

$$\tau\beta\,(\mathbb{N}, R, m) \models^{j} \psi$$

$\Leftrightarrow$ $\{$ definition of $\tau\beta$ $\}$

$$(\mathbb{N}, \mathrm{suc} : \mathbb{N} \to \mathbb{N}, m) \models^{j} \psi$$

$\Leftrightarrow$ $\{$ definition of $\models^{\mathcal{L}\mathcal{I}}$ $\}$

$$m(j) \models \psi$$

$\Leftrightarrow$ $\{$ definition of $\models^{\mathcal{L}\mathcal{I}}$, definition of σ $\}$

$$(\mathbb{N}, R, m) \models^{j} \sigma(\psi)$$

(b) $X\rho$,

$$\tau\beta\,(\mathbb{N}, R, m) \models^{j} X\rho$$

$\Leftrightarrow$ $\{$ definition of $\models^{\mathcal{L}\mathcal{I}}$ $\}$

$$\tau\beta\,(\mathbb{N}, R, m) \models^{j+1} \rho$$

$\Leftrightarrow$ $\{$ induction hypothesis $\}$

$$(\mathbb{N}, R, m) \models^{j+1} \sigma(\rho)$$

$\Leftrightarrow$ $\{$ R_{Next} defines the successor function $\}$

$$(\mathbb{N}, R, m) \models^{j} [Next]\,\sigma(\rho)$$

$$\Leftrightarrow \qquad \{ \text{ definition of } \sigma \ \}$$

$$(\mathbb{N}, R, m) \models^j \sigma(X\rho)$$

(c) The remaining cases are proved similarly.