Chapter 2
Optimal Localization of Firms in Hotelling Networks

Alberto Adrego Pinto and Telmo Parreira

2.1 Introduction

Hotelling [1] introduce a model of spatial competition in a city represented by a line segment, where a uniformly distributed continuum of consumers have to buy a homogeneous good. Consumers have to support the transportation costs when baying the good in one of the two firms of the city. In this framework, firms simultaneously choose their location and afterwards set their prices in order to maximize their profits. In [2], the Hotelling's line model is extended to networks, where the firm's location is taken exogenously, that is, the focus consists in studying the second stage of the two-stages game where the firms compete in prices. In this model, the edges are inhabited by consumers uniformly distributed, in the spirit of Hotelling's line model and Salop's [4] circular model.

The *Hotelling town* model consists of a network of *consumers* and *firms*. The consumers (*buyers*) are located along the *edges* (*roads*) of the network and the firms (*shops*) are located at the *vertices* (*nodes*) of the network. Every road has two vertices and at every vertex is located a single firm. The *degree* k of the vertex is given by number of incident edges. If the degree $k > 2$ then the vertex is a crossroad of k roads; if the degree $k = 2$ then the vertex is a junction between two roads; and if $k = 1$ the vertex is in the end of a no-exit road. Every consumer will buy one unit

A.A. Pinto
LIAAD-INESC TEC, Porto, Portugal

Department of Mathematics, Faculty of Sciences, University of Porto, Rua do Campo Alegre, 687, 4169-007, Porto, Portugal,
e-mail: aapinto@fc.up.pt

T. Parreira (✉)
Department of Mathematics, University of Minho, Campus de Gualtar, 4710-057 Braga, Portugal

LIAAD-INESC TEC, Porto, Portugal
e-mail: telmoparreira@hotmail.com

A.A. Pinto and D. Zilberman (eds.), *Modeling, Dynamics, Optimization and Bioeconomics I*, Springer Proceedings in Mathematics & Statistics 73,
DOI 10.1007/978-3-319-04849-9_2,

of the commodity to only one of the firms of the network and every firm will charge a price for the commodity that will be the same for all its customers.

A Hotelling town *price strategy* $\mathbf{P}$ consists of a vector whose coordinates are the prices p_i of each firm F_i. A consumer located at a point x of the network that decides to buy at firm F_i *spends* $E(x; i, \mathbf{P}) = p_i + t\, d(i, x)$, where p_i is the price charged by the firm F_i and $t\, d(i, x)$ is the *transportation cost* that is proportional t to the minimal distance $d(i, x)$ between the position i of the firm F_i and the position x of the consumer measured in the network. The *local firms* of a consumer located at a point x in a road $R_{i,j}$ with vertices i and j are F_i and F_j. In every road $R_{i,j}$, there is at most one consumer located at a point $\mathbf{x}_{i,j} \in R_{i,j}$ that is *indifferent* to in which local firm is going to buy his commodity, i.e. $E(x; i, \mathbf{P}) = E(x; j, \mathbf{P})$. A price strategy $\mathbf{P}$ determines a *local market structure* if every road $R_{i,j}$ has an indifferent consumer $\mathbf{x}_{i,j} \in R_{i,j}$.

We introduce the BLC condition that gives a bound for the different production costs of firms and for the different road lengths of the network in terms of the transportation cost, the minimal road length of the network and the maximum node degree (see Sect. 2.2). For the Hotelling town network satisfying the BLC condition, we exhibit the unique optimum Nash price equilibrium strategy $\mathbf{P}^*$ (see the proof in [3]).

We say that a price strategy has the *profit degree growth* property if the profits of the firms increase with the degree of the nodes. We introduce the DB condition that gives a bound for the different production costs of firms and for the different road lengths of the network in terms of the transportation cost, the minimal road length of the network and the maximum node degree (see Sect. 2.2). We show that for a Hotelling town network satisfying the DB and BLC conditions the Nash price equilibrium strategy has the profit degree growth property.

If all firms have the same production costs and all roads have the same length then we say that the Hotelling town is uniform. A uniform Hotelling town satisfies the BLC and DB conditions and so has a unique Nash price equilibrium with the profit degree growth property.

We show that a Hotelling town network satisfying the BLC condition and with all degrees of the nodes above 2 has the property that for small perturbations on the localization of the firms the Nash price equilibrium is unique and it is a small perturbation of $\mathbf{P}^*$ (see Sect. 2.3). Furthermore, the profit of a firm is optimal at the node for small perturbations on its own localization. Hence, the node localization of the firms is locally optimal. All the results presented in this chapter are proved in [3].

2.2 Nash Equilibrium Price Strategy

Given a price strategy $\mathbf{P}$, the consumer will choose to buy in the firm $F_{v(x,\mathbf{P})}$ that minimizes his expenditure in the Hotelling town

$$v(x, \mathbf{P}) = argmin_{i \in V} E(x; i, \mathbf{P}),$$

where V is the set of all vertices of the Hotelling town. Hence, for every firm F_i, the *market*

$$M(i, \mathbf{P}) = \{x : v(x, \mathbf{P}) = i\}$$

consists of all consumers that minimize their expenditure by opting to buy in firm F_i.

The *road market size* $l_{i,j}$ of a road $R_{i,j}$ is the Lebesgue measure (or length) of the road $R_{i,j}$, because the buyers are uniformly distributed along the roads, and the *market size* $S(i, \mathbf{P})$ of the firm F_i is the Lebesgue measure of $M(i, \mathbf{P})$. The Hotelling town *production cost* $\mathbf{C}$ is the vector whose coordinates are the production costs c_i of the firms F_i. The Hotelling town *profit* $\Pi(\mathbf{P}, \mathbf{C})$ is the vector whose coordinates

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i)\, S(i, \mathbf{P})$$

are the *profits* of the firms F_i. A price strategy $\mathbf{P}$ determines a *local market structure* if every consumer buys in a local firm

$$M(i, \mathbf{P}) \subset \bigcup_{j \in N_i} R_{i,j}.$$

Hence, for every road $R_{i,j}$ there is an *indifferent buyer* located at a distance

$$0 < x_{i,j} = (2\,t)^{-1}(p_j - p_i + t\, l_{i,j}) < l_{i,j} \tag{2.1}$$

of firm F_i. Thus, a price strategy $\mathbf{P}$ determines a local market structure if, and only if, $|p_i - p_j| < t\, l_{i,j}$ for every road $R_{i,j}$.

The firms F_i and F_j (or vertices i and j) are *neighbors* if there is a road $R_{i,j}$ with end nodes i and j. Let N_i the set of all vertices that are neighbors of the vertex i and, so, k_i is the cardinality of the set N_i that is equal to the degree of the vertex i. If the price strategy determines a local market structure then $S(i, \mathbf{P}) = \sum_{j \in N_i} x_{i,j}$ and

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i) \sum_{j \in N_i} x_{i,j} = (2\,t)^{-1}(p_i - c_i) \sum_{j \in N_i} (p_j - p_i + t\, l_{i,j}).$$

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$.

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\, S(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \geqslant (p_i' - c_i)\, S(i, \mathbf{P}_i'),$$

for all $i \in V$ and for all price strategies $\mathbf{P}_i'$ whose coordinates satisfy $p_i' \geqslant c_i$ and $p_j' = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

We denote by c_M (resp. c_m) the maximum (resp. minimum) production cost of the Hotelling town

$$c_M = \max\{c_i : i \in V\} \quad \text{and} \quad c_m = \min\{c_i : i \in V\}.$$

We denote by l_M (resp. l_m) the maximum (resp. minimum) road length of the Hotelling town

$$l_M = \max\{l_e : e \in E\} \quad \text{and} \quad l_m = \min\{l_e : e \in E\},$$

where E is the set of all edges of the Hotelling town. Let

$$\Delta(c) = c_M - c_m \quad \text{and} \quad \Delta(l) = l_M - l_m.$$

We denote by k_M be the maximum node degree of the Hotelling town

$$k_M = \max\{k_i : i \in V\}.$$

Definition 2.1. A Hotelling town satisfies the *bounded length and costs (BLC)* condition, if

$$\Delta(c) + t\,\Delta(l) \leqslant \frac{(t\,l_m - \Delta(c)/2)^2}{2\,t\,k_M\,l_M}. \tag{2.2}$$

The Hotelling town *admissible market size* **L** is the vector whose coordinates are the *admissible local firm market sizes*

$$L_i = \frac{t}{k_i} \sum_{j \in N_i} l_{i,j}.$$

The Hotelling town *neighboring market structure* **K** is the matrix whose coordinates are (1) $k_{i,j} = k_i^{-1}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and (2) $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j. Let **1** denote the identity matrix.

Theorem 2.1. *If a Hotelling town satisfies the BLC condition then there is a unique Hotelling town Nash equilibrium price strategy*

$$\boldsymbol{P}^* = \frac{1}{2}\left(\boldsymbol{1} - \frac{1}{2}\boldsymbol{K}\right)^{-1}(\boldsymbol{C} + \boldsymbol{L}) = \sum_{m=0}^{\infty} 2^{-(m+1)}\boldsymbol{K}^m\,(\boldsymbol{C} + \boldsymbol{L}). \tag{2.3}$$

We note that the Nash equilibrium price strategy for the Hotelling town satisfying the BLC condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other

firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.
We say that a price strategy $\mathbf{P}$ has the *profit degree growth* property if

$$k_i > k_j \Rightarrow \pi_i(\mathbf{P}, \mathbf{C}) > \pi_j(\mathbf{P}, \mathbf{C})$$

for every $i, j \in V$.

Lemma 2.1. *Let F_i be a firm located in a node of degree k_i and F_j a firm located in a node of degree k_j. Let $\bar{p}_i = p_i^* - c_i$ and $\bar{p}_j = p_j^* - c_j$ represent the unit profit of firm F_i and F_j, respectively. Then, $\pi_i^* > \pi_j^*$ if and only if*

$$\frac{k_i - k_j}{k_j} > \frac{\bar{p}_j^2 - \bar{p}_i^2}{\bar{p}_i^2}.$$

Definition 2.2. A Hotelling town network satisfies the *degree bounded lengths and costs (DB)* condition if

$$\Delta(c) + t\,\Delta(l) < \left(\sqrt{1 + 1/k_M} - 1\right)(t\,l_m - \Delta(c)/2)\,. \tag{2.4}$$

Theorem 2.2. *A Hotelling town network satisfying the BLC and DB conditions has the profit degree growth property.*

A *cost uniform* Hotelling town is a Hotelling town with $\Delta(c) = 0$. A *length uniform* Hotelling town is a Hotelling town with $\Delta(l) = 0$. A *uniform* Hotelling town is a Hotelling town with $\Delta(c) = \Delta(c) = 0$. We note that the degrees of the nodes can be different.

Remark 2.1. In a cost uniform Hotelling town, if $2\,k_M\,l_M\,\Delta(l) \leqslant l_m^2$ there is a unique network Nash price strategy; in a length uniform Hotelling town, if $2\,t\,k_M\,l_M\,\Delta(c) \leqslant (t\,l_M - \Delta(c)/2)^2$ there is a unique network Nash price strategy; and in a uniform Hotelling town there is a unique network Nash price strategy that satisfies the profit degree growth property

2.3 Local Stability

Consider that a firm F_i located a node i changes its location to a point y_i in a road $R_{i,j}$ at distance x for the node i. Let $\mathbf{P}(x; i, j)$ denote the Nash equilibrium price strategy taking in account the new localization of the firm F_i and let $\pi_i(x; i, j)$ denote the profit of firm F_i with respect to the price strategy $\mathbf{P}(x; i, j)$.

Definition 2.3. We say that a firm F_i is *node local stable* if there is $\epsilon_i > 0$ such that $\pi_i(0; i, j) > \pi_i(x; i, j)$ for every $0 < x < \epsilon_i$, with respect to the local optimal equilibrium price strategy. A Hotelling network is *firm position local stable* if every firm in the network is node stable.

We denote by k_M be the maximum node degree of the Hotelling town

$$k_m = \min\{k_i : i \in V\}.$$

Theorem 2.3. *A Hotelling town satisfying the BLC condition and with $k_m \geqslant 3$ is firm position local stable.*

Firms F_i with $k_i = 2$ are node local unstable, except for networks satisfying special symmetric properties. Firms F_i with $k_i = 3$ whose neighboring firms have nodes degree greater or equal to 3 are node local stable. Furthermore, firms F_i with $k_i \geqslant 4$ whose neighboring firms have nodes degree greater or equal to 2 are node local stable.

2.4 Conclusion

We presented a model of price competition in a network, extending the linear city presented by Hotelling to a network where firms are located at the nodes and consumers distributed along the edges. Under a condition on the production costs, road lengths and maximum node degree we found the unique pure Nash price strategy for which the Hotelling tow has a local market structure, i.e. the consumers prefer to buy at the local firms. Finally, we show that for small perturbations of its own localization, a Hotelling network where all nodes have degree greater that two has an optimal localization strategy consisting in all firms located at the nodes.

Acknowledgements We acknowledge the financial support of LIAAD-INESC TEC through 'Strategic Project—LA 14—2013–2014' with reference PEst-C/EEI/LA0014/2013, USP-UP project, IJUP, Faculty of Sciences, University of Porto, Calouste Gulbenkian Foundation, FEDER, POCI 2010 and COMPETE Programmes and Fundação para a Ciência e a Tecnologia (FCT) through Project "Dynamics and Applications", with reference PTDC/MAT/121107/2010.

References

1. Hotelling, H.: Stability in competition. Econ. J. **39**, 41–57 (1929)
2. Pinto, A.A., Parreira, T.: A hotelling-type network. In: Peixoto, M., Pinto, A.A., Rand, D. (eds.) Dynamics, Games and Science. I. Springer Proceedings in Mathematics, vol. 1, pp. 709–720. Springer, Berlin (2011)
3. Parreira, T., Pinto, A.A.: Local market structure in a hotelling town. Submited Paper, 2013
4. Salop, S.: Monopolistic competition with outside goods. Bell J. Econ. **10**, 141–156 (1979)